Generalized New Extended Weibull Distribution with Real Life Application

Zubair Ahmad, Department of Statistics, Quaid-e-Azam University, Islamabad, Pakistan
email: z.ferry21@gmail.com

Syed Muhammad Hamid, Department of Statistics, Quaid-e-Azam University, Islamabad, Pakistan
email: smhamid@stat.qau.edu.pk

Zawar Hussain, Department of Statistics, Quaid-e-Azam University, Islamabad, Pakistan
email: zhlangah@yahoo.com

Abstract- In the present study, a new lifetime distribution is introduced by considering a serial system of two component parts, with one component following a Rayleigh distribution and other part following a new extended Weibull distribution. The new model may be named as generalized new extended Weibull distribution and is able to model lifetime data with increasing, unimodal or modified unimodal shaped failure rates. A brief derivation of the mathematical properties including moments, generation of random numbers and densities of the order statistics will be provided. The maximum likelihood estimates of the unknown parameters will be obtained. The proposed distribution will be illustrated by analyzing a real data set and its goodness of fit result will be compared with some of the prominent extensions of the Weibull distribution.

Keywords: Weibull distribution, Order statistics, Moment generating function, Maximum likelihood estimation

I. INTRODUCTION

In the practice of reliability theory, the probability distributions are frequently used as time to failure distributions. In same scenario, the usefulness of reliability model significantly depends on the success in choosing appropriate probability distribution of the phenomenon under consideration. During the past couple of years, a particular group of the statistical distributions such as, Weibull, Rayleigh and exponential distributions were used frequently in modeling lifetime data. But, in the practice, we find that most of these distributions are not flexible enough to accommodate different phenomena of nature. For example, the exponential distribution is able to model lifetime data with constant hazard rates only, the Rayleigh distribution has increasing hazard rates. However, the Weibull distribution is much flexible than the exponential and Rayleigh distributions and is able to model lifetime phenomena with monotonic failure rates (i.e. increasing, decreasing or constant hazard rates). For this reason, the researchers have worked on the development and extension of these distributions to have a more flexible and more suited model for modeling lifetime data in practice. The traditional Weibull model by Waloddi Weibull [25] is one of the most frequently used lifetime distributions for modeling lifetime phenomena. The practical applications of the Weibull distribution can be found in ([11], [15], [17], [18], [19], [20], [21] and [23]). But, the Weibull distribution is inappropriate to use for modeling data with non-monotonic failure rates (i.e. unimodal, modified unimodal or bathtub shaped failure rates). To improve the flexibility of the Weibull distribution, it has been modified by a number of authors to obtain non-monotonic hazard rates, see for example, ([1], [2], [3], [4], [9], [10], [12], [13], [14], [16], [22], [24] and [26]). Recently, Ahmad and Hussain [6] proposed a very interesting modified form of Weibull distribution with survival function (SF) given by
The model defined in (1) is capable of modeling data with increasing, unimodal, modified unimodal or bathtub failure rates. The three parameters extension of the Weibull model proposed by Ahmad and Hussain [7] has increasing, unimodal or modified unimodal failure rates defined by the SF

\[ S(z) = e^{- \left( \frac{\beta \cdot z}{\sigma} \right)^{\alpha}}, \quad z > 0. \]  

(1)

Another three parameters very flexible Weibull (VFW) model proposed by Ahmad and Hussain [8] has increasing, unimodal or modified unimodal failure rates having SF given by

\[ S(z) = e^{- \left( \beta \cdot z^{\alpha} \right)^{\frac{1}{\beta^{2}}}}, \quad z > 0. \]  

(2)

In the recent time, Ahmad and Hussain [5] proposed a new extension of the Weibull distribution called new extended Weibull (NEx-W) distribution has the cumulative hazard function (CHF) given by

\[ H_{\text{NEx-W}}(z) = e^{\left( \frac{\beta \cdot z^{\alpha}}{\sigma} \right)^{\frac{1}{\beta^{2}}}}, \quad z, \alpha, \beta, \sigma > 0. \]  

(3)

The expression of CHF of the Rayleigh distribution is given by

\[ H_{R}(z) = \theta z^{2}, \quad z, \theta > 0. \]  

(5)

It is a very useful technique to combine two different survival functions and generate a new function as

\[ S(z) = \eta S_{1}(z) + (1-\eta) S_{2}(z), \]

where $0 < \eta < 1$, it is better known as a mixture of models/distributions, or

\[ S(z) = \eta S_{1}(z) + \gamma S_{2}(z), \]  

(6)

with parameters $\eta, \gamma > 0$. One can also generate a new function by combining two cumulative hazard functions. In term of CHF, the cumulative distribution function (CDF) can be written as

\[ G(z) = 1 - e^{-H(z)}, \]  

(7)

where the CHF denoted by $H(z)$ fulfills the following two conditions

i. \[ H(z) \text{ is non-negative as well as increasing function of } z. \]

ii. \[ \lim_{z \to 0} H(z) \to 0 \text{ and } \lim_{z \to \infty} H(z) \to \infty. \]

In this paper, the CHF’s of the NEx-W and Rayleigh distributions are combined to develop a new function as

\[ H_{\text{proposed}}(z) = H_{\text{Rayleigh}}(z) + H_{\text{NEx-W}}(z). \]  

(8)
Using (9), in (7), one may get the CDF of the proposed model which is titled as generalized new extended Weibull (GNE\textsuperscript{x}-W) distribution. The proposed model is very flexible and is able to model lifetime data with increasing, unimodal or modified unimodal failure rates. The models expressed in (1)-(3), belongs to class defined in (9).

The rest of article is organized in the following manner: Section II offers definition, motivation and usefulness of the new model. Section III derives statistical properties of the proposed distribution such as quantile function, moments, and order statistics. The moment generating and factorial moment generating functions are derived in section IV and V, respectively. Section VI considers estimation of the unknown parameters. A real data set is analyzed in Section VII, and the result is compared with other well-known existing lifetime distributions. Finally, section VIII concludes the article.

II. GENERALIZED NEW EXTENDED WEIBULL DISTRIBUTION

This section considers the definition and motivation of the proposed distribution.

A. Definition

The proposed distribution is defined by the following CDF

\[
G(z) = 1 - e^{-\theta z^2 - e^{\frac{\beta z^{\alpha}}{\sigma^2}}} \quad z, \alpha, \beta, \sigma, \theta > 0.
\]  

(10)

The density function of the proposed distribution is given by

\[
g(z) = \frac{2\theta z + \left(\alpha \beta z^{\alpha-1} + \frac{2\sigma}{z^3}\right)e^{\frac{\beta z^{\alpha}}{\sigma^2}}}{\left(\frac{1}{\theta} - \frac{\beta z^{\alpha}}{\sigma^2}ight)^2 + \frac{2\sigma}{z^3}} e^{-\theta z^2 - e^{\frac{\beta z^{\alpha}}{\sigma^2}}}.
\]  

(11)

B. Motivation and Interpretation

The SF of the proposed model is given by

\[
S(z) = e^{-\theta z^2 - e^{\frac{\beta z^{\alpha}}{\sigma^2}}},
\]  

(12)

and, the failure rate function is

\[
h(z) = 2\theta z + \left(\alpha \beta z^{\alpha-1} + \frac{2\sigma}{z^3}\right)e^{\frac{\beta z^{\alpha}}{\sigma^2}}.
\]  

(13)

The figure 1 & 2 displays the HF of the proposed model for \( \sigma = 0.3 \) and different values of \( \alpha, \beta \) and \( \theta \), respectively.
Figure 1: HFs of the GNEx-W distribution, for selected values of the parameters.

Figure 2: HFs of the GNEx-W distribution, for selected values of the parameters.
It may be interpreted as a serial scheme with two independent components, one of which has the NEx-W distribution with parameters $\alpha$, $\beta$ and $\sigma$, and the other has the Rayleigh distribution with parameter $\theta$. The propose of the newly proposed distribution is to model lifetime phenomena with increasing, unimodal or modified unimodal shaped hazard rates which are quite common in reliability and bio-medical analysis. For example, the increasing failure rate function is very useful for modeling the lifetime of a machine component. Whereas, the unimodal and modified unimodal hazard rates are very suitable for modeling lifetime cycle of breast cancer patients and very helpful in determining the time period having maximum failure rate.

III. Basic Mathematical Properties

The section contains the basic mathematical properties of the GNEx-W distribution.

A. The Quantile Function

The expression for the $q^{th}$ quantile $z_q$ of the GNEx-W distribution is given by

$$\beta z_q^\alpha - \frac{\sigma}{z_q^\alpha} - \log \{-\log (1 - q) + \theta z^2\} = 0. \quad (14)$$

As it is quite clear that the expression for the quantile function of the proposed distribution is not in closed form. Therefore, the closed form solution of the quantile function can be obtained using computer software.

B. The Moments

Let $Z$ has the GNEx-W model with parameters $(\alpha, \beta, \sigma, \theta)$, then the $r^{th}$ moments of $Z$ can be derived as

$$\mu_r = \int_0^\infty z^r g(z; \alpha, \beta, \sigma, \theta) dz,$$

$$\mu_r = \int_0^\infty z^{r+1} \left\{2 \theta z + \left(\alpha \beta z^{\alpha-1} + \frac{2 \sigma}{z}\right) e^{\frac{\beta z^\alpha - \sigma}{\theta z}}\right\} e^{-\theta z^2} dz,$$

$$\mu_r = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^\infty z^{r+i} \left\{2 \theta z + \left(\alpha \beta z^{\alpha-1} + \frac{2 \sigma}{z}\right) e^{\frac{\beta z^\alpha - \sigma}{\theta z}}\right\} \left(\frac{\beta z^\alpha - \sigma}{\theta z}\right)^i e^{-\theta z^2} dz,$$

$$\mu_r = \theta \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^\infty z^{r+i} \left(\frac{\beta z^\alpha - \sigma}{\theta z}\right)^i e^{-\theta z^2} dz + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^\infty z^{r+i} \left(\alpha \beta z^{\alpha-1} + \frac{2 \sigma}{z}\right) \left(\frac{\beta z^\alpha - \sigma}{\theta z}\right)^i e^{-\theta z^2} dz.$$

Finally, the following expression is observed

$$\mu_r = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^i}{i!} \left(\sigma i \right)^k \left(\beta i \right)^j \omega W_i + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^i}{i!} \left(i + 1\right)^{i+k} \beta^j \left(\frac{\alpha \beta}{2} W_2 + \sigma W_3\right). \quad (15)$$
Where,

\[
W_1 = \frac{\Gamma\left(\frac{r + j\alpha - 2\kappa}{2}\right) + 1}{\theta},
\]

\[
W_2 = \frac{\Gamma\left(\frac{r + \alpha(\text{ }j + 1) - 2\kappa}{2}\right)}{\theta},
\]

and

\[
W_3 = \frac{\Gamma\left(\frac{r + j\alpha - 2(k + 1)}{2}\right)}{\theta}.
\]

C. Densities of the Order Statistics

Let \( Z_1, Z_2, \ldots, Z_k \) be independently and identically distributed (i.i.d) random variables taken from GNEX-W distribution with parameters \((\alpha, \beta, \sigma, \theta)\), in such a way that \( Z_{(1:k)} \leq Z_{(2:k)} \leq \ldots \leq Z_{(k:k)} \). So, the density function of \( Z_{(i:k)}, i = 1, 2, 3, \ldots, k \) is

\[
g_{ik}(z) = \frac{1}{\text{Beta}(i, k-i+1)} g(z, \Phi)[G(z, \Phi)]^{i-1} \left[1 - G(z, \Phi)\right]^{k-i}.
\]  

(16)

Here, the densities for the \( 1^{st} \) order statistic as \( Z_{(1)} = \min(Z_1, Z_2, \ldots, Z_k) \) and for \( k^{th} \) order statistic as \( Z_{(k)} = \max(Z_1, Z_2, \ldots, Z_k) \) are derived in (17) and (18), respectively.

\[
g_{1k}(z) = k g(z)\left[1 - G(z)\right]^{k-1}.
\]

(17)

Also, density for the maximum order statistics is

\[
g_{kk}(z) = k g(z)\left[G(z)\right]^{k-1},
\]

\[
g_{kk}(z) = k \left[2\theta z + \left(\alpha\beta z^{a-1} + \frac{2\sigma}{z^a}\right)e^{\left(\beta z - \frac{\sigma}{z^a}\right)}\right] e^{-\theta z} \left[1 - e^{-\theta z}\right]^{k-1}.
\]

(18)
IV. MOMENT GENERATING FUNCTION

Let $Z$ has the GNEx-W distribution with parameters $(\alpha, \beta, \sigma, \theta)$, then the moment generating function (MGF) of $Z$ is derived as

$$M_z(t) = E(e^{zt})$$

$$M_z(t) = \int_{0}^{\infty} e^{zt} g(z; \alpha, \beta, \sigma, \theta) dz,$$

$$M_z(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r.$$  \hspace{1cm} (19)

Using (15), in (19), one may have the solution of the MGF of NMEx-W distribution.

V. FACTORIAL MOMENT GENERATING FUNCTION

Let $Z$ has the GNEx-W distribution with parameters $(\alpha, \beta, \sigma, \theta)$, then the factorial moment generating function (FMGF) of $Z$ can be derived as

$$H_0(\delta) = E\left((1 + \delta)^z\right)$$

$$H_0(\delta) = E\left(e^{z \ln(1+\delta)}\right),$$

$$H_0(\delta) = \int_{0}^{\infty} e^{z \ln(1+\delta)} g(z; \alpha, \beta, \sigma) dz,$$

$$H_0(\delta) = \sum_{r=0}^{\infty} \frac{(\ln'(1+\delta))}{r!} \mu_r.$$  \hspace{1cm} (20)

Using (15), in (20), one may obtain the FMGF of NMEx-W distribution.

VI. ESTIMATION

This section of the paper derives the maximum likelihood estimates of the model parameters. Let $Z_1, Z_2, \ldots, Z_k$ be selected randomly from GNEx-W distribution with parameters $(\alpha, \beta, \sigma, \theta)$, then the log-likelihood function of this sample is

$$\ln L = \sum_{i=1}^{k} \log \left[ 2\theta z_i + \left( \alpha \beta z_i^{-1} + \frac{2\sigma}{z_i} \right) e^{\left( \beta^2 z_i^{-2} - \frac{\beta}{z_i} \right)} \right] - \theta \sum_{i=1}^{k} z_i^2 - \frac{k}{2} \sum_{i=1}^{k} e^{\left( \beta^2 z_i^{-2} - \frac{\beta}{z_i} \right)}.$$  \hspace{1cm} (21)

By obtaining the partial derivatives of the expression in (21) on behalf of parameters, and then equating the results equal to zero, one may get
From the expressions provided in (22)-(25), it is clear that these equations are not in closed forms, and cannot be solved manually. Therefore, the solution of these equations can be obtained numerically by using the iterative techniques such as the Newton-Raphson algorithm. The “SANN” algorithm is used in R language to obtain the numerical estimates of the parameters.

**VII. REAL LIFE APPLICATION**

To illustrate the substantial advances of the proposed model, an example to a real data set is analyzed. The goodness of fit result of the GNEx-W distribution is compared with that of five other well-known lifetime models such as NEx-W, Weibull (W), Flexible Weibull extension (FWEx), generalized power Weibull (GPW) and Kumaraswamy generalized power Weibull (Ku-GPW) distributions. The analytical tools including Kolmogorov–Smirnov (K-S) test statistic, Akaike’s Information Criterion (AIC), Hannan-Quinn information criterion (HQIC), corrected Akaike information criterion (CAIC) and log likelihood \(-2l(., z)\) are considered for deciding the goodness of fit result.

Taking decision using these analytical measures, it is proved that the proposed model delivers greater distributional flexibility as compared to other competitive models. The data set picked form Ahmad and Hussain [5], having the single fibers of 20 mm, with a sample of size 63.
TABLE 1: A sample of 63 single fibers of 20 MM.


TABLE 2: Summary of the single fibers data.

<table>
<thead>
<tr>
<th>Min.</th>
<th>1st Quartile.</th>
<th>Median</th>
<th>Mean</th>
<th>3rd Quartile.</th>
<th>Max.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.901</td>
<td>2.554</td>
<td>2.996</td>
<td>3.059</td>
<td>3.422</td>
<td>5.020</td>
</tr>
</tbody>
</table>

The selected analytical criteria of the proposed model, NEx-W, W, FWEx, GPW and Ku-GPW distributions are provided in table 3.

TABLE 3: Goodness of fit results of GNEx-W, NEx-W, W, FWEx, GPW and KU-GPW.

<table>
<thead>
<tr>
<th>Dist.</th>
<th>Max. Likelihood Estimates</th>
<th>-2logl</th>
<th>AIC</th>
<th>CAIC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>GNEx-W</td>
<td>( \hat{\alpha} =0.125, \hat{\beta} =1.011, \hat{\sigma} =9.865, \hat{\theta} =0.932 )</td>
<td>52.56</td>
<td>114.54</td>
<td>113.32</td>
<td>115.76</td>
</tr>
<tr>
<td>NEx-W</td>
<td>( \hat{\alpha} =0.254, \hat{\beta} =1.711, \hat{\sigma} =23.65 )</td>
<td>56.47</td>
<td>118.94</td>
<td>119.35</td>
<td>121.47</td>
</tr>
<tr>
<td>W</td>
<td>( \hat{\alpha} =5.049, \hat{\beta} =3.315 )</td>
<td>61.95</td>
<td>127.91</td>
<td>128.11</td>
<td>129.44</td>
</tr>
<tr>
<td>FWEx</td>
<td>( \hat{\beta} = 0.307, \hat{\sigma} = 1.396 )</td>
<td>61.59</td>
<td>127.19</td>
<td>127.39</td>
<td>128.88</td>
</tr>
<tr>
<td>GPW</td>
<td>( \hat{\alpha} =3.151, \hat{\beta} =19.824, \hat{\sigma} =179.363 )</td>
<td>69.27</td>
<td>144.54</td>
<td>144.94</td>
<td>147.07</td>
</tr>
<tr>
<td>Ku-GPW</td>
<td>( \hat{\alpha} =40.07, \hat{\beta} =1.40, \hat{\alpha} =0.67, \hat{\beta} =2.21, \hat{\sigma} =0.46 )</td>
<td>56.94</td>
<td>122.69</td>
<td>123.74</td>
<td>126.90</td>
</tr>
</tbody>
</table>

VIII. Conclusion

A new lifetime distribution, based on new extended Weibull and Rayleigh distributions, has been introduced and its mathematical properties are studied. The idea is to combine two components in a serial system, to propose a new lifetime distribution capable of modeling data with increasing, unimodal or modified unimodal shaped hazard rates. By analyzing a real data set, it showed that the generalized new extended Weibull distribution fits certain well-known data set better than the Weibull model and other well-known extensions of the Weibull distribution.

Acknowledgments

We would like to thank the editor and referees for their comments and suggestions which improved the presentation of the paper.

References


