Adjusted Adomian Decomposition Method for Solving Emden-Fowler Equations of Various Order

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Abstract—In this search, we consider the singular initial value problems of order n. We given the new operator for studying this problems and we given illustrations this method by some examples for comparing the exact solutions by the solution of this method, the operator we find developing for three parts of problems. The numerical results showed that our present method is less time consuming and easier to use than other methods.

Keywords: Adjusted Adomian decomposition method, Emden–Fowler equation, initial value problem.

1. INTRODUCTION

We consider the Emden - Fowler Equations of the type [11, 12,17]

\[ y'' + \frac{h}{x} y' + \frac{k}{x^2} y + f(x, y) = 0, \quad (1) \]

where \( f(x, y) \) is a real functions of \( x \) and \( y \), respectively where \( h \geq 2 \) and \( k \geq 0 \) are called the shape factor. These issues by and large emerge every now and again in numerous ranges of science and building, for instance, fluid mechanics, quantum mechanics, idea control, chemical reactor hypothesis, optimal design, response dispersion handle, geophysics, and so forth. The singular behavior that occurs at \( x = 0 \) is the main difficulty of (1).

The Emden-Fowler equation were subjected to a considerable size of investigation, both numerically and analytically. A variety of useful methods were used to obtain exact and approximate solutions as well. Examples of the methods that were applied are the Adomian decomposition method [9,18,19,21,24,27,30], the variational iteration method [13,22,28,29,32,33], the homotopy perturbation method [31], the rational Legendre pseudospectral approach [23], and other methods as well [11,12].

The Adomian decomposition method (ADM) [1,2,3,5,6,7,10,14,15,16,20], proposed by Adomian toward the start of 1980s, has gotten huge consideration in the previous two decades. Adomian declares that the decay technique gives an efficient and computationally helpful strategy for producing estimated arrangement to the wide class of conditions. It has been utilized by Adomian and numerous different creators to research a huge assortment of scientific and physical issues. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral differential equations as well. The Adomian decomposition method has proved to be effective and reliable in handling both linear and nonlinear equations, and gives successive components of the solution without any need for an ad hoc transformation or perturbation technique.

We aim in this work to establish various kinds of Emden-Fowler type equations of various order. Our approach depends mainly on using different orders of different operators involved in the Emden-Fowler sense given in (2). Our next goal of this work is to apply Adomian Decomposition Strategy to handle the derived Emden-Fowler type equations of various kind. Several numerical examples, with specified initial conditions will be examined to handle the singular point that exist in each equation.
2. CONSTRUCTING EMDEN-FOWLER KINDS EQUATIONS

It is interesting to note that the Emden - Fowler equation (1) was derived by using the equation

\[ x^{-n} \frac{d}{dx} x^{n-1} \frac{d}{dx} x^m y(x) + f(x, y) = 0, \]  
(2)

Where \( h = n + 1 \), and \( k = n - 1 \).

To derive the Emden-Fowler kind equations of different order, we use the sense of (2) and set

\[ x^{-n} \frac{d}{dx} x^{n-1} \frac{d^m}{dx^m} x(y) + f(x, y) = 0, \]  
(3)

where \( n \geq 1 \). To determine such different order equations we set \( m \) to different values.

2.1 First Kind for \( m = 1 \)

Substituting for \( m = 1 \) in (3) gives

\[ y'' + \frac{n + 1}{x} y' + \frac{n - 1}{x^2} y + f(x, y) = 0. \]  
(4)

2.2 Second Kind for \( m = 2 \)

Substituting for \( m = 2 \) in (3) gives

\[ y''' + \frac{n + 2}{x} y'' + \frac{2(n - 1)}{x^2} y' + f(x, y) = 0. \]  
(5)

2.3 Third Kind for \( m = 3 \)

Substituting for \( m = 3 \) in (3) gives

\[ y^{(4)} + \frac{n + 3}{x} y''' + \frac{3(n - 1)}{x^2} y'' + f(x, y) = 0. \]  
(6)

2.4 Fourth Kind for \( m = 4 \)

Substituting for \( m = 4 \) in (3) gives

\[ y^{(5)} + \frac{n + 4}{x} y^{(4)} + \frac{4(n - 1)}{x^2} y''' + f(x, y) = 0. \]  
(7)

\[ y^{(m+1)} + \frac{n + m}{x} y^{(m)} + \frac{m(n - 1)}{x^2} y^{(m-1)} + f(x, y) = 0. \]  
(8)

3. MODIFIED ADOMIAN DECOMPOSITION METHOD

Consider the singular initial value problem of \( m + 1 \) order ordinary differential equation in the sense of (8) as following :

\[ y^{(m+1)} + \frac{n + m}{x} y^m + \frac{m(n - 1)}{x^2} y^{(m-1)} + f(x, y) = 0, \]  
(9)

\[ y(0) = a_0, y'(0) = a_1, \ldots, y^{(m)}(0) = a_m. \]

Where \( a_0, a_1, \ldots, a_m \) are given constants and \( f(x, y) \) is a real function.

We rewrite (9) in the form

\[ Ly = -f(x, y), \]  
(10)

Where the differential operator \( L \) is defined by
The inverse operator $L^{-1}$ is therefore considered a $m+1$ fold integral operator, as below

$$L^{-1}(.) = x^{-1} \int_0^x \cdots \int_0^{x^{m-n}} x^n \, dx \, dx.$$  \hspace{1cm} (12)

Applying $L^{-1}$ on (10) we find

$$L^{-1}(Ly) = -L^{-1}f(x,y) \quad \text{Which gives}$$

$$y(x) = \Phi(x) - L^{-1}f(x,y).$$  \hspace{1cm} (13)

Such that $L(\Phi(x)) = 0$.

The Adomian decomposition method introduces the solution $y(x)$ by an infinite series of components

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$  \hspace{1cm} (14)

and

$$f(x,y) = \sum_{n=0}^{\infty} A_n,$$  \hspace{1cm} (15)

where the components $y_n(x)$ of the solution will be determined recurrently. Specific algorithms were seen in [25,26] formulate Adomian polynomials. The following algorithm:

$$A_0 = f(u_0),$$

$$A_1 = f'(u_0)u_1,$$

$$A_2 = f''(u_0)u_2 + \frac{1}{2} f'''(u_0)u_1^2,$$

$$A_3 = f'''(u_0)u_3 + f''(u_0)u_1u_2 + \frac{1}{3!} f^{(iv)}(u_0)u_1^3,$$  \hspace{1cm} (16)

can be used to construct Adomian polynomials, when $f(u)$ is a nonlinear function.

By substituting (14) and (15) into (13),

$$\sum_{n=0}^{\infty} y_n(x) = \Phi(x) - L^{-1} \sum_{n=0}^{\infty} A_n.$$  \hspace{1cm} (17)

Through using Adomian decomposition method, the components $y_n(x)$ can be determined as

$$y_0 = \Phi(x),$$

$$y_{n+1}(x) = -L^{-1}A_n \quad n \geq 0,$$  \hspace{1cm} (18)

Which gives

$$y_0 = \Phi(x),$$

$$y_1 = -L^{-1}A_0.$$
\[ y_2 = -L^{-1}A_1, \]
\[ y_3 = -L^{-1}A_2, \]
\[ \vdots \]

From (16) and (19), we can determine the components \( y_n(x) \), and hence the series solution of \( y(x) \) in (14) can be immediately obtained.

For numerical purposes, the \( n \)-term approximate
\[ \Psi_n = \sum_{n=0}^{\infty} y_n(x), \]

(20)
can be used to approximate the exact solution.

4. NUMERICAL EXAMPLES

Example 1. We consider the Emden–Fowler type equation:
\[ y'' + \frac{3}{x} y' + \frac{1}{x^2} y = 9e^x + 7xe^x + y, \]
\[ y(0) = 0, \quad y'(0) = 0. \]

With exact solution \( x^2e^x \).

Eq.(21) can be written as
\[ Ly = 9e^x + 7xe^x + y. \]

(22)

Where
\[ L(\cdot) = x^{-2} \frac{d}{dx} x \frac{d}{dx} x(\cdot). \]

And operating
\[ L^{-1}(\cdot) = x^{-1} \int_0^x x^{-1} \int_0^x x^2 (\cdot) dx dx. \]

On both sides of (22), and using the initial conditions at \( x = 0 \), yields
\[ L^{-1}(Ly) = L^{-1}(9e^x + 7xe^x + y), \]
\[ L^{-1}(Ly) = L^{-1}(9e^x + 7xe^x) + L^{-1}(y), \]
\[ y(x) = 0 + L^{-1}(9e^x + 7xe^x) + L^{-1}(y), \]
\[ \sum_{m=0}^{\infty} y_n(x) = 0 + L^{-1}(9e^x + 7xe^x) + L^{-1}(y), \]
\[ y_0 = L^{-1}(9e^x + 7xe^x), \]
\[ y_{n+1} = L^{-1}(A_n), \quad n \geq 0. \]

Using (25), the first several calculated solution components are
\[ y_0 = 9 \left( \frac{x^2}{9} + \frac{x^4}{16} + \frac{x^6}{50} + \frac{x^8}{216} + \frac{x^{10}}{1176} + \cdots \right) + 7 \left( \frac{x^3}{16} + \frac{x^5}{25} + \frac{x^7}{72} + \frac{x^9}{294} + \cdots \right), \]
\[ y_1 = L^{-1}(y_0) = 9 \left( \frac{x^4}{225} + \frac{x^6}{576} + \frac{x^8}{2450} + \cdots \right) + 7 \left( \frac{x^5}{576} + \frac{x^7}{1225} + \frac{x^9}{4608} + \cdots \right). \]
\[ y_2 = L^{-1}(y_1) = 9 \left( \frac{x^6}{11025} + \frac{x^7}{36864} + \cdots \right) + 7 \left( \frac{x^7}{36864} + \cdots \right). \]

\[ y(x) = y_0 + y_1 + y_2 + \cdots, \]

\[ y(x) = x^2 + x^3 + \frac{1}{2} x^4 + \frac{1}{6} x^5 + \frac{1}{24} x^6 + \cdots \quad (26) \]

Note that, the Taylor series of exact solution \( y(x) = x^2 e^x \) is as below

\[ y(x) = x^2 + x^3 + \frac{1}{2!} x^4 + \frac{1}{3!} x^5 + \frac{1}{4!} x^6 + \cdots \]

Example 2. We consider the Emden–Fowler type equation:

\[ y^{(4)} + \frac{7}{x} y''' + \frac{9}{x^2} y'' = 300 + x^4 - y, \quad (27) \]

\[ y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0. \]

With exact solution is \( x^4 \).

Eq.(27) can be written as

\[ Ly = 300 + x^4 - y, \quad (28) \]

Where

\[ L(.) = x^{-4} \frac{d}{dx} x^3 \frac{d^3}{dx^3} x(.), \]

And operating

\[ L^{-1}(.) = x^{-1} \int_0^x \int_0^x \int_0^x \int_0^x x^4 (.) dx dx dx dx. \]

On both sides of (28), and using the initial conditions at \( x = 0 \), yields

\[ L^{-1}(Ly) = L^{-1} (300 + x^4 - y), \]

\[ L^{-1}(Ly) = L^{-1} (300 + x^4) - L^{-1}(y), \]

\[ y(x) = L^{-1} (300 + x^4) - L^{-1}(y). \quad (29) \]

Substituting the decomposition series \( y_n(x) \) for \( y(x) \) into (29) gives

\[ \sum_{n=0}^{\infty} y_n(x) = 0 + L^{-1} (300 + x^4) - L^{-1}(y), \quad (30) \]

\[ y_0 = L^{-1} (300 + x^4), \]

\[ y_{n+1} = -L^{-1}(A_n), \quad n \geq 0. \quad (31) \]

Using (31), the first several calculated solution components are

\[ y_0 = x^4 + \frac{1}{4536} x^8, \]

\[ y_1 = -L^{-1}(y_0) = -\frac{1}{4536} x^8 - \frac{1}{101189088} x^{12}, \]

\[ y_2 = -L^{-1}(y_1) = \frac{1}{101189088} x^{12} + \frac{1}{7018475144000} x^{16}, \]

\[ y_3 = -L^{-1}(y_2) = -\frac{1}{7018475143680} x^{16} - \frac{1}{117615606457789400} x^{20}, \]
Other components can be evaluated in a similar manner. It is easily observed that the noise terms appear in $y_0$ and $y_1$ with opposite signs. Canceling these noise terms from $y_0$ gives the exact solution

$$y(x) = x^4.$$  \hspace{1cm} (32)

Example 3. We consider the Emden–Fowler type equation:

$$y'' + \frac{5}{x}y' + \frac{3}{x^2}y = 15 - x^4 + y^2,$$  \hspace{1cm} (33)

$$y(0) = 0, \quad y'(0) = 0, \quad .$$

With exact solution $= x^2$.

Eq.(33) can be written as

$$L y = 15 - x^4 + y^2.$$  \hspace{1cm} (34)

Where

$$L(.) = x^{-4} \frac{d}{dx} x^4 \frac{d}{dx} x(.) .$$

And operating

$$L^{-1}(.) = x^{-1} \int_0^x x^{-3} \int_0^x x^4 (.) dx dx .$$

On both sides of (34), and using the initial conditions at $x = 0$, yields

$$L^{-1}(Ly) = L^{-1}(15 - x^4 + y^2),$$

$$L^{-1}(Ly) = L^{-1}(15 - x^4) + L^{-1}(y^2),$$

$$y(x) = L^{-1}(15 - x^4) + L^{-1}(y^2).$$  \hspace{1cm} (35)

Substituting the decomposition series $y_n(x)$ for $y(x)$ into (34) gives

$$\sum_{n=0}^{\infty} y_n(x) = 0 + L^{-1}(15 - x^4) + L^{-1}(y^2),$$  \hspace{1cm} (36)

$$y_0 = L^{-1}(15 - x^4),$$

$$y_{n+1} = L^{-1}(A_n), \quad n \geq 0.$$  \hspace{1cm} (37)

$$A_0 = y_0^2 ,$$

$$A_1 = 2y_0y_1 ,$$

$$A_2 = y_1^2 + 2y_0y_2 ,$$

$$A_3 = 2y_1y_2 + 2y_0y_3 .$$  \hspace{1cm} (38)

Using (38), the first several calculated solution components are

$$y_0 = x^2 - \frac{x^6}{63},$$

$$y_1 = L^{-1}(y_0^2) = \frac{x^6}{63} - \frac{2}{9009} x^{10} + \frac{1}{1012095} x^{14},$$
\[ y_2 = L^{-1}(2y_0y_1) = \frac{2}{9009}x^{10} - \frac{538}{144729585}x^{14} + \frac{1306}{57747104415}x^{18} - \frac{2}{3663141375}x^{22}, \]

\[ y(x) = x^2 - \frac{79}{28945917}x^{14} + \frac{1306}{57747104415}x^{18} - \frac{2}{3663141375}x^{22} + \ldots \quad (39) \]

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Example 4. We consider the Emden–Fowler type equation:

\[ y''' + \frac{4}{x}y'' + \frac{2}{x^2}y' = 3(12 + 30x^3 + 9x^6)y, \quad (40) \]
\[
\begin{align*}
y(0) &= 1, \quad y'(0) = 0, \quad y''(0) = 0. \\
\text{With exact solution }&= e^{x^3}.
\end{align*}
\]

Eq.(40) can be written as
\[
Ly = 3(12 + 30x^3 + 9x^6)y. 
\tag{41}
\]

Where
\[
L(.) = x^{-2} \frac{d}{dx} x \frac{d^2}{dx^2} x(.).
\]

And operating
\[
L^{-1}(.) = x^{-1} \int_0^x \int_0^x \int_0^x x^2(.) dx dx dx.
\]

On both sides of (41), and using the initial conditions at \(x = 0\), yields
\[
L^{-1}(Ly) = L^{-1}(3(12 + 30x^3 + 9x^6)y),
\]
\[
y(x) = 1 + L^{-1}(3(12 + 30x^3 + 9x^6)y). 
\tag{42}
\]

Substituting the decomposition series \(y_n(x)\) for \(y(x)\) into (42) gives
\[
\sum_{n=0}^{\infty} y_n(x) = 1 + L^{-1}(3(12 + 30x^3 + 9x^6)y),
\tag{43}
\]
\[
y_0 = 1,
\]
\[
y_{n+1} = L^{-1}(3(12 + 30x^3 + 9x^6)A_n), \quad n \geq 0. 
\tag{44}
\]

Using (44), the first several calculated solution components are
\[
y_0 = 1,
\]
\[
y_1 = L^{-1}(3(12 + 30x^3 + 9x^6)y_0 = x^3 + \frac{5}{14}x^6 + \frac{1}{30}x^9,
\]
\[
y_2 = L^{-1}(3(12 + 30x^3 + 9x^6)y_1 = \frac{1}{7}x^6 + \frac{8}{63}x^9 + \frac{44}{1365}x^{12} + \frac{59}{50400}x^{15}
\]
\[
+ \frac{1}{6840}x^{18},
\]
\[
y_3 = L^{-1}(3(12 + 30x^3 + 9x^6)y_2 = \frac{2}{315}x^9 + \frac{61}{6552}x^{12} + \frac{1069}{234000}x^{15}
\]
\[
+ \frac{81176}{784274400}x^{18} + \frac{569877}{5636318688}x^{21} + \frac{4763}{1532160000}x^{24} + \frac{1}{5171040}x^{27},
\]
\[
y(x) = y_0 + y_1 + y_2 + y_3 + \ldots
\]
\[
y(x) = 1 + x^3 + \frac{1}{2}x^6 + \frac{1}{6}x^9 + \frac{1361}{32760}x^{12} + \frac{2089}{364000}x^{15} + \frac{2514593}{2128744800}x^{18}
\]
\[
+ \frac{569877}{5636318688}x^{21} + \frac{4763}{1532160000}x^{24} + \frac{1}{5171040}x^{27} + \ldots
\tag{45}
\]
Table 2. Comparison of numerical errors.

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5. CONCLUSION

In this work, we have established various kinds of Emden-Fowler type equations of various order. We used the Adomian decomposition method for treating linear and nonlinear problems to illustrate our analysis. The obtained results validate the reliability and rapid convergence of the ADM. We demonstrated that the decomposition procedure is quite efficient for determining solution in closed form by using initial conditions or boundary conditions. Our present methods avoid the tedious work needed by traditional techniques.
6. REFERENCES