On Normal and Soft Normal Groups under Multisets and Soft Multiset Context

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Abstract - In this paper, the concepts of normal submultigroups and soft normal multigroups with some of their related algebraic structures were developed. We established that intersection, union of any two normal submultigroups is also normal submultigroup of a given multigroup. The inverse of any normal submultigroup of a multigroup is a normal submultigroup and for any normal submultigroup, the root (support) set is a normal subgroup of the underlying group. We then showed that under the isomorphism function between any two groups, the image of a normal submultigroup under the isomorphism is a normal submultigroup and the inverse image of a normal submultigroup under the isomorphism is a normal submultigroup. Finally, we defined operations on soft normal multigroups such as intersection, union, AND, OR operations and discovered that such operations are closed under soft normal multigroups.

Key Words: multiset, soft multiset, multigroup, submultigroup, soft multigroups, normal submultigroup, soft normal multigroups, isomorphism function.

I. INTRODUCTION

Theory of Multisets is an important generalization of classical set theory which has emerged by violating a basic property of classical sets that an element can belong to a set only once. The term multiset (mset for short) as Knuth [1] notes, first suggested by [2] in a private communication. Owing to its aptness, it has replaced a variety of terms viz. list, heap, bunch, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonymity with mset. This set theory has various applications in mathematics and computer science, overview of which can be obtained in [3]. Many authors like [4, 5, 6, 7] have studied the properties of msets. Molodtsov [8] initiated a novel concept of soft set theory, which is a completely new approach for modeling vagueness and uncertainties. Soft set theory has a rich potential for applications in several directions, few of which has been shown by [8]. Maji et al [9, 10] worked on some mathematical aspects of soft sets. Some authors have also generalized the notion of msets in the settings of fuzzy sets [30] and soft sets [8] to form fuzzy multiset[23], multisoft sets [11] and soft multisets [12, 13].
Msets and soft msets have been applied in multiple type of scenario’s such as in information retrieval on the web, multicriteria decision making, knowledge representation in data based systems, biological systems and membrane computing as will be found in [14, 15, 16]. Several authors have introduced the notion of groups in fuzzy sets, intuitionistic L-fuzzy sets, soft sets, fuzzy soft sets, multisets, fuzzy multisets, intuitionistic fuzzy multi sets, and soft multi set settings to form fuzzy groups [29], intuitionistic L-fuzzy groups [28], soft groups [27], fuzzy soft groups [26], multigroups [20], fuzzy multigroups [25], intuitionistic fuzzy multigroups [24] and soft multi groups [20]. Yohanna and Daniel [17, 18] defined a symmetric group under multiset context and presents a study of the classical group theory in the context of msets. In this paper, we developed the concepts of normal submgroups and soft normal mgroups with some of their related algebraic structures.

II. PRELIMINARIES

**Definition 2.1** [7]. An mset $M$ drawn from the set $X$ is represented by a Count Function $C_M$ defined as $C_M: X \rightarrow \mathbb{N}$, where $\mathbb{N}$ represents the set of non-negative integers. Here $C_M(x)$ is the number of occurrence of the element $x$ in the mset $M$. The presentation of the mset $M$ drawn from $X = \{x_1, x_2, \ldots, x_n\}$ will be denoted by $M = \{m_1, m_2, \ldots, m_n\}$ where $m_i$ is the number of occurrences of the element $x_i \in X$. We denote the set of all finite msets drawn from $X$ by $\mathbb{M}(X)$.

**Definition 2.2** [21]. Let $M_1, M_2 \in \mathbb{M}(X)$. Then $M_1$ is said to be subset of $M_2$ if $C_{M_1}(x) \leq C_{M_2}(x), \forall x \in X$. This relation is denoted by $M_1 \subseteq M_2$, $M_2$ is said to be equal to $M_2$ if $C_{M_1}(x) = C_{M_2}(x), \forall x \in X$. It is denoted by $M_1 = M_2$.

**Definition 2.3** [7]. Let $\{M_i \in \mathbb{M}(X) | i \in I\}$ be a nonempty family of msets drawn from the set $X$. Then

(a) Their Intersection, denoted by $\bigcap_{i \in I} M_i$ is defined:
$$C_{\bigcap_{i \in I} M_i}(x) = \bigwedge_{i \in I} C_{M_i}(x), \forall x \in X$$

where $\bigwedge_{i \in I} C_{M_i}(x) = \min \{C_{M_i}(x)\}$

(b) Their Union, denoted by $\bigcup_{i \in I} M_i$ is defined:
$$C_{\bigcup_{i \in I} M_i}(x) = \bigvee_{i \in I} C_{M_i}(x), \forall x \in X$$

where $\bigvee_{i \in I} C_{M_i}(x) = \max \{C_{M_i}(x)\}$

(c) Their addition denoted $\Sigma_{i \in I} M_i$ is defined:
$$C_{\Sigma_{i \in I} M_i}(x) = \Sigma_{i \in I} C_{M_i}(x), \forall x \in X$$

(d) The Complement of any mset $M_i$ in $\mathbb{M}(X)$, denoted by $M_i^C$ is defined:
$$C_{M_i^C}(x) = C_Z(x) - C_{M_i}(x), \forall x \in X$$

where $Z = \bigcup_{i \in I} M_i$ for all $M_i \in \mathbb{M}(X)$

**Definition 2.4** [19]. Let $X$ and $Y$ be two nonempty sets and
$f : X \rightarrow Y$ be a mapping. Then

i. The image of a mset $M \in \mathcal{M}(X)$ under the mapping $f$ is denoted by $f(M)$, where

$$C_{f(M)}(y) = \begin{cases} 
V_f(x)=y \in C_M(x) & \text{if } f^{-1}(y) \neq \phi \\
0 & \text{otherwise}
\end{cases}$$

ii. The inverse image of a mset $N \in \mathcal{M}(Y)$ under the mapping $f$ is denoted by $f^{-1}(N)$, where

$$C_{f^{-1}(N)}(x) = C_N[f(x)]$$

Definition 2.5 [19]. A mset containing only one element $x$ occurring $n$ times is called a singleton mset and it is denoted by $n_x$.

Soft Msets

Definition 2.6 [22]. Let $M \in \mathcal{M}(X)$. The set $M^0 = \{x \in X : C_M(x) > 0\}$, is called the support set or root set of $M$.

Definition 2.7 [22]. The set of all submsets of $M \in \mathcal{M}(X)$, denoted by $P^+(M)$, is called the power set of the mset $M$.

Definition 2.8 [22]. Let $M \in \mathcal{M}(X)$, $E$ be a set of parameters and $A \subseteq E$. Then a pair $(F, A)$ is called a soft mset over $M$ where $F$ is a mapping given by $F : A \rightarrow P^+(M)$.

Definition 2.9 [22]. Let $(F, A)$, $(G, B)$ be two soft msets over $M \in \mathcal{M}(X)$. Then $(F, A)$ is said to be sub soft mset of $(G, B)$ if $A \subseteq B$ and $F(\alpha) \subseteq G(\alpha), \forall \alpha \in A$.

Definition 2.10 [22]. Let $(F, A)$, $(G, B)$ be two soft msets over $M \in \mathcal{M}(X)$. Then

i. Their restricted intersection, denoted by $(F, A) \cap (G, B) = (F \cap G, A \cap B)$, is defined by $(F \cap G)(\alpha) = [F(\alpha) \cap G(\alpha)], \forall \alpha \in (A \cap B)$.

ii. Their extended intersection, denoted by $(F, A) \cap_E (G, B) = (H, C)$ is defined by

$$H(\alpha) = \begin{cases} 
F(\alpha) & \text{if } \alpha \in A - B; \\
G(\alpha) & \text{if } \alpha \in B - A; \\
F(\alpha) \cap G(\alpha) & \text{if } \alpha \in A \cap B.
\end{cases}$$

Where $C = A \cup B$ and all $\alpha \in C$.

iii. Their restricted union, denoted by $(H, C) = (F, A) \cup (G, B)$ is defined by

$$H(\alpha) = F(\alpha) \cup G(\alpha), \forall \alpha \in C.$$ 

Where $C = A \cap B$.

iv. Their extended union, denoted by $(F, A) \cup_E (G, B) = (F \cup G, A \cup B)$, is defined by
\( \forall \alpha \in (A \cup B), \)

\[
(F \cup G)(\alpha) = \begin{cases} 
F(\alpha) & \text{if } \alpha \in (A - B); \\
G(\alpha) & \text{if } \alpha \in (B - A); \\
F(\alpha) \cup G(\alpha) & \text{if } \alpha \in (A \cap B); 
\end{cases}
\]

v. Their AND, denoted by \((F, A) \wedge (G, B) = (F \wedge G, A \times B)\), is defined by

\[
(F \wedge G)(\alpha, \beta) = [F(\alpha) \cap G(\beta)], \forall (\alpha, \beta) \in (A \times B).
\]

vi. The complement of \((F, A)\), denoted by \((F, A)^c = (F^c, A)\), is defined by;

\[
F^c(\alpha) = M - F(\alpha), \forall \alpha \in A,
\]

where

\[
C_{M - F_A}(x) = C_M(x) - C_{F_A}(x), \forall x \in M^a.
\]

**Definition 2.11** [19]. Let \( A \in \mathbb{M}(X) \). Then the inverse, \( A^{-1} \) is defined such that:

\[
C_{A^{-1}}(x) = C_A(x^{-1}).
\]

**Definition 2.12** [19]. Let \( X \) be a group. A mset \( G \in \mathbb{M}(X) \) is said to be a mgroup if the Count Function of \( G, C_G \) satisfies the following two conditions.

\[
C_G(xy) \geq [C_G(x) \wedge C_G(y)], \forall x, y \in X;
\]

\[
C_G(x^{-1}) \geq C_G(x), \forall x \in X.
\]

We denote the set of all mgroups over \( X \) by \( MG(X) \).

**Theorem 2.13** [19]. Let \( G \in MG(X) \). Then

i. \( C_G(e) \geq C_G(x), \forall x \in X \);

ii. \( C_G(x^n) \geq C_G(x), \forall x \in X \);

iii. \( C_G(x^{-1}) = C_G(x), \forall x \in X \);

iv. \( G = G^{-1} \).

**Proposition 2.14.** Let \( G \in MG(X) \). Then \( C_G(x^n y^n) \geq [C_G(x) \wedge C_G(y)], \forall x, y \in X, \forall n \geq 1 \) such that \( n \in \mathbb{N} \).

Proof.

From Theorem 2.13 (ii) clearly,

\[
C_G(x^n) \geq C_G(x), \forall x \in X \tag{1}
\]

\[
C_G(y^n) \geq C_G(y), \forall y \in X \tag{2}
\]

From equations (1) and (2) we have;

\[
C_G(x^n y^n) \geq [C_G(x^n) \wedge C_G(y^n)] \geq [C_G(x) \wedge C_G(y)]
\]

\[
\Rightarrow C_G(x^n y^n) \geq [C_G(x) \wedge C_G(y)], \forall x, y \in X, \forall n \geq 1 \text{ such that } n \in \mathbb{N}.
\]
Theorem 2.15 [19]. Let $X, Y$ be two groups and $f: X \rightarrow Y$ be a homomorphism of groups.
If $G \in MG(X)$, then $f(G) \in MG(Y)$ and
if $H \in MG(Y)$, then $f^{-1}(H) \in MG(X)$.

Definition 2.16 [19]. Let $G \in MG(X)$. Then $G_\phi$ and $G^\phi$ are defined as follows:
$G_\phi = \{ x \in X; C_G(x) = C_G(\phi) \}$
and $G^\phi = \{ x \in X; C_G(x) > 0 \}$.

Theorem 2.17 [19]. Let $G \in MG(X)$. Then $G_\phi$ and $G^\phi$ are subgroups of $X$.

Theorem 2.18 [19]. If $\{ G_i; i \in I \}$ be a family of mgroups over a group $X$, then their intersection $\bigcap_{i \in I} G_i$ is a mgroup over $X$.

Definition 2.19 [19]. Let $G_1, G_2 \in MG(X)$. Then $G_1$ is said to be a subgroup of $G_2$ if $G_1 \subseteq G_2$.

Definition 2.20 [20]. Let $X$ be a group, $M$ be a mgroup over $X$ and $A \subseteq E$ be a set of parameters. A soft mset $(F, A)$ drawn from $M$ is said to be a soft mgroup over $M$ if $F(\alpha)$ is a subgroup of $M, \forall \alpha \in A$.

Definition 2.21. If $(F_1, A)$ and $(F_2, B)$ are two soft mgroups over a mgroup $M$, then "$(F_1, A)$ AND $(F_2, B)$", denoted by $(F_1, A) \wedge (F_2, B)$, is defined
$F_5(\alpha_1, \alpha_2) = F_1(\alpha_1) \cap F_2(\alpha_2)$
$\forall (\alpha_1, \alpha_2) \in A \times B$.

Definition 2.22. If $(F_1, A)$ and $(F_2, B)$ are two soft mgroups over a mgroup $M$, then "$(F_1, A)$ OR $(F_2, B)$", denoted by $(F_1, A) \vee (F_2, B)$, is defined
$F_5(\alpha_1, \alpha_2) = F_1(\alpha_1) \cup F_2(\alpha_2)$
$\forall (\alpha_1, \alpha_2) \in A \times B$.

Theorem 2.23 [20]. Let $M$ be a mgroup over a group $X$ and $(F_1, A), (F_2, B)$ be two soft mgroups over $M$. Then their restricted intersection $(F_1, A) \wedge (F_2, B)$ is a soft mgroup over $M$.

Theorem 2.24 [20]. If $(F_1, A)$ and $(F_2, B)$ be two soft mgroups over $M$, then $(F_1, A) \wedge (F_2, B)$ is a soft mgroup over $M$.

Definition 2.25 [20]. Let $(F_1, A_1)$ and $(F_2, A_2)$ be two soft mgroups over $M$. Then $(F_1, A_1)$ is said to be a Sub soft mgroup of $(F_2, A_2)$, denoted by $(F_1, A_1) \subseteq (F_2, A_2)$ if $A_1 \subseteq A_2$ and $F_1(\alpha)$ is a subgroup of $F_2(\alpha), \forall \alpha \in A_1$. 
Proposition 2.26. Let \((F_1, A_1), (F_2, A_2)\) and \((F_3, A_3)\) be three mgroups over \(M\). If \((F_1, A_1) \subseteq (F_2, A_2)\) and \((F_2, A_2) \subseteq (F_3, A_3)\). Then \((F_1, A_1) \subseteq (F_3, A_3)\).

Proof.

\((F_1, A_1) \subseteq (F_2, A_2) \Rightarrow A_1 \subseteq A_2 \text{ and } \forall \alpha \in A_1, F_1(\alpha) \text{ is a subgroup of } F_2(\alpha) \) \hspace{1cm} (1)

\((F_2, A_2) \subseteq (F_3, A_3) \Rightarrow A_2 \subseteq A_3 \text{ and } \forall \beta \in A_2, F_2(\beta) \text{ is a subgroup of } F_3(\beta) \) \hspace{1cm} (2)

But \((\subseteq \text{ is transitive relation})\)

i.e. \(A_1 \subseteq A_2 \subseteq A_3 \Rightarrow A_1 \subseteq A_3\) \hspace{1cm} (3)

If \(F_1(\alpha)\) is a subgroup of \(F_2(\alpha)\)

Then \(F_1(\alpha) \subseteq F_2(\alpha)\) \hspace{1cm} (4)

If \(F_2(\alpha)\) is a subgroup of \(F_3(\alpha)\)

Then \(F_2(\alpha) \subseteq F_3(\alpha)\) \hspace{1cm} (5)

From (4) and (5) clearly \(F_1(\alpha) \subseteq F_3(\alpha)\) \hspace{1cm} (6)

Comparing (3) and (6) we have;

\((F_1, A_1) \subseteq (F_3, A_3)\).

III. RESULTS AND DISCUSSION

In this section, we defined the concepts of normal subgroups and soft normal mgroups and deduce some of their related results. Throughout the chapter, unless otherwise stated, \(X\) will be assumed to be a group, \(M\) be a mgroup over \(X\) and \(E, K\) be the set of parameters,

\[ A_1 \subseteq E_1, B_1 \subseteq K, t \in \Delta. \]

Definition 3.1. A subgroup \(H\) of a mgroup \(M \in MG(\mathbb{X})\) is said to be a normal subgroup iff for any \(k \in H^*, c_h(x^{-1}hx) \geq c_b(h), \forall x \in M^*\).

We denote the set of all normal subgroups of a mgroup \(M \in MG(\mathbb{X})\) by \(\mathcal{M}(M)\) and the normality of \(H\) over \(M \in MG(\mathbb{X})\) by \(M \triangleleft M\).

Example (1) let \(X = \{e, x, y, z\}\) be Klein’s 4-group, \(M = \{e, e, e, e, x, y, y, y, z, z\}\) be a mgroup over \(X\), let consider the following subgroups of \(M\):

\[ M_1 = \{e, e, e, x, y, y, y, z, z\}, \]
\[ M_2 = \{e, e, x, x, y, y, y, z, z\}, \]
\[ M_3 = \{e, e, e, x, x, y, y, z, z\}. \]

Clearly \(M_1, M_2, M_3\) are normal subgroups of \(M\).

Example (2) the set \(X = Z_5 = \{0, 1, 2, 3, 4\}\) is a group with respect to addition and \(M = \{0, 0, 0, 0, 1, 1, 1, 2, 2, 3, 3, 4, 4\}\) is a mgroup over \(X\). The subgroups \(M_1, M_2, M_3, M_4\) given by:

\[ M_1 = \{0, 0, 0, 0, 1, 1, 1, 2, 2, 3, 3, 4, 4\}, M_2 = \{0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4\}, M_3 = \{0, 0, 0, 1, 1, 2, 2, 3, 3, 4, 4\} \]

are clearly normal subgroups of \(M\).
Proposition 3.2. Let \( M_1 \) and \( M_2 \) be subgroups of a mgroup \( M \in MG(X) \). Such that \( M_1, M_2 \in \Delta P(M) \), then

(i) \( M_1 \cap M_2 \in \Delta P(M) \),

(ii) \( M_1 \cup M_2 \in \Delta P(M) \).

Proof.

(i) Since \( M_1, M_2 \in \Delta P(M) \) we have:
\[
C_{M_1}(xy) \succeq C_{M_1}(x) \wedge C_{M_1}(y), \forall x, y \in X,
\]
\[
C_{M_2}(xy) \succeq C_{M_2}(x) \wedge C_{M_2}(y), \forall x, y \in X.
\]
\[
\therefore C_{M_1 \cap M_2}(xy) = \bigwedge \left[ C_{M_1}(x) \wedge C_{M_1}(y) \right] = C_{M_1}(x) \wedge C_{M_2}(y), \forall x, y \in X.
\]
\[
C_{M_1 \cap M_2}(x^{-1}) = C_{M_1}(x^{-1}) \wedge C_{M_2}(x^{-1}) = C_{M_2}(x) = C_{M_2}(x), \forall x \in X.
\]
Thus, \( M_1 \cap M_2 \in MG(X) \) \( \Box \) (1)

(ii) Since \( M_1, M_2 \subseteq M \), then \( M_1 \cap M_2 \subseteq M \) \( \Box \) (2)

Since \( M_1, M_2 \in \Delta P(M) \), we have
\[
C_{M_2}(x^{-1}y_1x), \forall x \in M^*, y_1 \in M_1^*;
\]
\[
C_{M_2}(x^{-1}y_2x), \forall x \in M^*, y_2 \in M_2^*.
\]

Now
\[
C_{M_1 \cap M_2}(x^{-1}yx) = \bigwedge \left[ C_{M_1}(x^{-1}yx), C_{M_2}(x^{-1}yx) \right] = \bigwedge \left[ C_{M_2}(y), C_{M_2}(y) \right] \forall x \in M^*, y \in (M_1 \cap M_2)^* = M_1^* \cap M_2^*
\]

Since \( C_{M_2}(x^{-1}y_1x) \succeq C_{M_2}(y_1), \forall x \in M^*, y_1 \in M_1^* \), \( \Box \) (3)

Therefore, \( C_{M_2}(x^{-1}y_2x) \succeq C_{M_2}(y_2), \forall x \in M^*, y_2 \in M_2^* \), \( \Box \) (4)

Now
\[
C_{M_1 \cap M_2}(x^{-1}yx) = \bigwedge \left[ C_{M_1}(y), C_{M_2}(y) \right] \forall x \in X^*, y \in M_1 \cap M_2
\]

Thus, \( C_{M_1 \cap M_2}(x^{-1}yx) \succeq C_{M_2}(y), \forall x \in M^*, y \in M_1 \cap M_2 \), \( \Box \) (5)

From (1), (2), (3), (4) and (5) we have, \( M_1 \cap M_2 \in \Delta P(M) \).
\[ C_{M_2}(xy) \geq C_{M_2}(x) \land C_{M_2}(y), C_{M_2}(x^{-1}) = C_{M_2}(x) \ \forall \ x, y \in X. \]

\[ C_{M_2M_2}(xy) = \lor \{ C_{M_2}(x) \lor C_{M_2}(y) \} \geq \lor \{ [C_{M_2}(x) \land C_{M_2}(y)] \lor [C_{M_2}(x) \land C_{M_2}(y)] \} = [C_{M_2}(x) \land C_{M_2}(y)] \lor [C_{M_2}(y) \land C_{M_2}(y)] = [C_{M_2}(x) \lor C_{M_2}(y)] \lor [C_{M_2}(y) \lor C_{M_2}(y)] = C_{M_2M_2}(x) \land C_{M_2M_2}(y) \] (1)

\[ C_{M_2M_2}(x^{-1}) = C_{M_2}(x^{-1}) \lor C_{M_2}(x^{-1}) = C_{M_2}(x) \lor C_{M_2}(x) = C_{M_2M_2}(x) \] (2)

\[ M_2 \cup M_2 \in MG(X) \] (3)

But \[ M_2 \cup M_2 \subseteq M \] (4)

Since \[ M_2, M_2 \in \Delta \varphi(M) \], we have
\[ C_{M_2}(x^{-1}y_1x), \forall \ x \in M^*, y_1 \in M_1^*; \]
\[ C_{M_2}(x^{-1}y_2x) \geq C_{M_2}(y_2), \forall \ x \in M^*, y_2 \in M_2^*; \]

Now
\[ C_{M_2M_2}(x^{-1}yx) = \lor \{ C_{M_2}(x^{-1}yx), C_{M_2}(x^{-1}yx) \} \geq \lor \{ [C_{M_2}(x^2) \lor C_{M_2}(y)] \lor [C_{M_2}(x^2) \lor C_{M_2}(y)] \} \]

Since \[ C_{M_2}(x^{-1}y_1x) \geq C_{M_2}(y_1) \land C_{M_2}(x^{-1}y_2x) \geq C_{M_2}(y_2) \]

\[ C_{M_2M_2}(x^{-1}yx) \geq C_{M_2M_2}(y), \forall \ x \in M^*, y \in M_1 \cup M_2 \] (5)

From (1), (2), (3), (4) and (5) we have, \[ M_1 \cup M_2 \in \Delta \varphi(M). \]

Remark: Let \[ \{ M_i : M_i \in \Delta \varphi(M), i \in \Delta \} \] then \[ \bigcap_{i \in \Delta} M_i \in \Delta \varphi(M) \]

and \[ \bigcup_{i \in \Delta} M_i \in \Delta \varphi(M) \]

Proposition 3.3. If \[ H \in \Delta \varphi(M) \] then \[ H^* \] is a normal subgroup of \[ M^* \].

Proof.

Since \[ H \subseteq M \Rightarrow H^* \subseteq M^* \] (see [4])

Thus, \[ H^* \] is a subgroup of \[ M^* \] (Theorem 2.17)

Since \[ H \in \Delta \varphi(M) \], then for any \[ x \in M^* \] and \[ k \in H^* \]
\[ C_{H^*}(x^{-1}hx) \geq C_{M^*}(k) > 0 \]

\[ C_{H^*}(x^{-1}hx) \geq 0 \]

\[ (x^{-1}hx) \in H^* \]

\[ H^* \] is a normal subgroup of \[ M^* \].

Proposition 3.4. Let \[ X \] be a group and \[ M \in MG(X) \]. If \[ M \in \Delta \varphi(M) \], then \[ M^{-1} \in \Delta \varphi(M) \].

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Proof.

Since $M \in \Delta \Phi (M)$, then we have,
$$C_M(x^{-1}yx) \supseteq C_M(y), \forall \ x \in X \text{ and } y \in M^*$$
Also
$$C_M^{-1}(xy) = C_M[(xy)^{-1}] = C_M(xy) \supseteq C_M(x) \wedge C_M(y)$$
$$= C_M(x^{-1}) \wedge C_M(y^{-1})$$
$$= C_M^{-1}(x) \wedge C_M^{-1}(y)$$
$$\therefore C_M^{-1}(xy) \supseteq C_M^{-1}(x^{-1}) \wedge C_M^{-1}(y^{-1})$$
$$C_M^{-1}(x^{-1}) = C_M[(x^{-1})^{-1}] = C_M(x)$$
$$= C_M(x^{-1}) = C_M^{-1}(x)$$
$$\therefore C_M^{-1}(x^{-1}) = C_M^{-1}(x) \quad (1)$$

Thus,
$$M^{-1} \in MG(X) \quad (3)$$
$$C_M^{-1}(x^{-1}yx) = C_M[(x^{-1}yx)^{-1}] = C_M(x^{-1}yx) \supseteq C_M(y), \forall \ x \in X \text{ and } y \in M^*$$
$$\therefore C_M^{-1}(x^{-1}yx) \supseteq C_M^{-1}(y)$$
$$\therefore C_M^{-1}(x^{-1}yx) \supseteq C_M^{-1}(y) \quad (4)$$

Hence from (1), (2), (3) and (4) we have $M^{-1} \in \Delta \Phi (M)$

**Proposition 3.5.** Let $X, Y$ be two groups and $f : X \rightarrow Y$ be an isomorphism of groups.

Suppose $M_1 \in MG(X), M_2 \in MG(Y)$ and $f(M_1) \subseteq M_2$

(i) if $H \in \Delta \Phi (M_1)$, then $f(H) \in \Delta \Phi (M_2)$

(ii) if $H \in \Delta \Phi (M_2)$, then $f^{-1}(H) \in \Delta \Phi (M_1)$

Proof.

(i) Let $H \in \Delta \Phi (M_1)$. Clearly $H$ is a subgroup of $M_1$ (by definition). We show that $f(H)$ is a subgroup of $M_2$.

$$f(H) = \bigcup_{f(z) \in H} C_M(z)$$

But $f$ is 1-1 and onto (by hypothesis), thus
$$f^{-1}(z) = \{ z \}$$

In particular,
$$f^{-1}(xy) = C_M(x) = C_M(f^{-1}(xy)) = C_M(f^{-1}(x)f^{-1}(y)) \quad (1)$$

But $f$ is an isomorphism. But
$$C_M(f^{-1}(x)f^{-1}(y)) \supseteq C_M(f^{-1}(x)) \wedge C_M(f^{-1}(y)) \quad (2)$$

Since $H$ is subgroup of $M_1$. But
$$C_M(f^{-1}(x)) \wedge C_M(f^{-1}(y)) = C_{(f^{-1})^{-1}}(x) \wedge C_{(f^{-1})^{-1}}(y) \quad (3)$$

(by definition 2.4(ii) and)
Thus using (1), (2), (3) and (4), we have:

\[ C_{f^{-1}g^{-1}(\mathcal{H})}(x) \cap C_{g^{-1}f^{-1}(\mathcal{H})}(y) = C_{f^{-1}g^{-1}(\mathcal{H})}(x) \cap C_{f^{-1}g^{-1}(\mathcal{H})}(y) \]  

(4)

Thus using (1), (2), (3) and (4), we have:

\[ C_{f^{-1}g^{-1}(\mathcal{H})}(x) \geq C_{f^{-1}g^{-1}(\mathcal{H})}(x) \cap C_{f^{-1}g^{-1}(\mathcal{H})}(y) \]

(5)

But \( f^{-1}(x^{-1}) = \emptyset \) (by definition 3.2.4(i))

Thus from (5) and (8) we have \( f(H) \in MG(Y) \)

(9)

But \( f(M_1) \subseteq M_2 \). Since \( H \subseteq M_1 \), then \( f(H) \subseteq M_2 \)

Hence \( f(H) \) is a subgroup of \( M_2 \)

(10)

Now we show that \( f(H) \subseteq M_2 \), i.e.

\[ C_{f^{-1}g^{-1}(\mathcal{H})}(x^{-1}y) \geq C_{f^{-1}g^{-1}(\mathcal{H})}(y) \]

for all \( x \in M_2 \) and \( y \in f(H) \)

(11)

Now \( f(x^{-1}y) = \bigvee_{f(x)=e} x^{-1}y \cap C_{H}(x) = C_{H}(f^{-1}(x^{-1}y)) \)

(12)

Hence, using (10) and (14) we deduce that \( f(H) \in MG(M_2) \)

(ii) Let \( H \) be a subgroup of \( M_2 \). We show that \( f^{-1}(H) \) is a subgroup of \( M_1 \).

But \( C_{f^{-1}g^{-1}(\mathcal{H})}(x) = C_{H}(f(x)) \) (by definition)

\[ \geq C_{H}(f(x)) \cap C_{H}(f(y)) \] (by definition)

\[ = C_{f^{-1}g^{-1}(\mathcal{H})}(x) \cap C_{f^{-1}g^{-1}(\mathcal{H})}(y) \]

(1)
\[ C_{f^{-1}(H)}(x^{-1}) = C_H[f(x^{-1})] \quad \text{(by definition)} \]
\[ = C_H \left[ (f(x))^{-1} \right] \quad \text{($f$ is an isomorphism)} \]
\[ = C_H[f(x)](H \text{ is a subgroup}) \]
\[ = C_{f^{-1}(H)}(x) \quad \text{(by definition)} \]

Thus \[ C_{f^{-1}(H)}(x^{-1}) = C_{f^{-1}(H)}(x) \quad \text{(2)} \]

Now for all \( x \in M \) and \( y \in [f^{-1}(H)]^* \) we have
\[ C_{f^{-1}(H)}(x y x^{-1}) = C_H[(f(x))^{-1} f(y)](f \text{ is an isomorphism}) \]
\[ \geq C_H[f(y)](H \in A \mathbb{P}(M_2) \text{ by hypothesis}) \]
\[ = C_{f^{-1}(H)}(y) \quad \text{(by definition)} \]

\[ C_{f^{-1}(H)}(x^{-1} y x) \geq C_{f^{-1}(H)}(y) \quad \text{(3)} \]

Hence from (1), (2) & (3) we have \( f^{-1}(H) \) is a normal subgroup of \( M_1 \).

**Definition 3.6.** Let \( X \) be a group, \( M \in MG(X) \) and \( A \subseteq \mathbb{R} \) be a set of parameters. A soft mgrou (\( F, A \)) drawn from \( M \) is said to be a soft normal mgrou over \( M \) if
\[ F(\alpha) \in A \mathbb{P}(M) \quad \text{for all} \quad \alpha \in A. \]

**Example (1)** let \( X = \{e, x, y, z\} \) be Klein's 4-group,
Then \( M = \{e, e, e, x, x, y, y, y, z, z\} \) is a mgrou over \( X \).
Let \( A = \{\alpha_1, \alpha_2, \alpha_3 \} \) and \( F: A \rightarrow P^*(M) \) be defined by
\[ F(\alpha_1) = \{e, e, e, e, x, x, y, y, y, z, z\}, \]
\[ F(\alpha_2) = \{e, e, e, e, x, x, y, y, y, z, z\}, \]
\[ F(\alpha_3) = \{e, e, e, e, x, x, y, y, y, z, z\}. \]
Clearly, \( F(\alpha_i), i = 1,2,3 \) is a normal subgroup of \( M \in MG(X) \)
and
\[ (F, A) = \{\{\alpha_1, \{e, e, e, x, x, y, y, y, z, z\}\}, \{\alpha_2, \{e, e, e, x, x, y, y, y, z, z\}\}, \{\alpha_3, \{e, e, e, x, x, y, y, y, z, z\}\}\} \]
is a soft normal mgrou over \( M \).

**Example (2)** the set \( X = Z_6 = \{0, 1, 2, 3, 4\} \) is a group with respect to addition and
\[ M = \{0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 4\} \]
is a mgrou over \( Z_6 \). Let \( A = \{\alpha_1, \alpha_2, \alpha_3\} \).
and \( F: A \rightarrow P^*(M) \) be defined:
\[ F(\alpha_1) = \{0, 0, 0, 1, 2, 3, 4\}, \]
\[ F(\alpha_2) = \{0, 0, 1, 2, 3, 4\}, \]
\[ F(\alpha_3) = \{0, 0, 1, 1, 2, 2, 3, 3, 4, 4\}. \]
Clearly, \( F(\alpha_i), i = 1,2,3 \) is a normal subgroup of \( M \in MG(X) \)
and \( (F, A) = \{\{\alpha_1, \{0, 0, 0, 0, 0, 1, 2, 3, 4\}\}, \{\alpha_2, \{0, 0, 1, 2, 3, 4\}\}, \{\alpha_3, \{0, 0, 1, 1, 2, 2, 3, 3, 4, 4\}\}\} \]
is a soft normal mgrou

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Definition 3.7. If $(F_1, A)$ and $(F_2, B)$ are two soft normal mgroups over a mgroup $M$, then “$(F_1, A)$ AND $(F_2, B)$”, denoted by $(F_1, A) \land (F_2, B)$, is defined
$$(F_1, A) \land (F_2, B) = (F_2, A \times B),$$
where $F_2(\alpha_1, \alpha_2) = F_1(\alpha_1) \cap F_2(\alpha_2) \forall (\alpha_1, \alpha_2) \in A \times B.$

Definition 3.8. If $(F_1, A)$ and $(F_2, B)$ are two soft normal mgroups over a mgroup $M$, then “$(F_1, A)$ OR $(F_2, B)$”, denoted by $(F_1, A) \lor (F_2, B)$, is defined
$$(F_1, A) \lor (F_2, B) = (F_2, A \times B),$$
where $F_2(\alpha_1, \alpha_2) = F_1(\alpha_1) \cup F_2(\alpha_2) \forall (\alpha_1, \alpha_2) \in A \times B.$

Definition 3.9. If $(F_1, A)$ and $(F_2, B)$ are two soft normal mgroups over a mgroup $M$, then the intersection of $(F_1, A)$ and $(F_2, B)$, denoted by “$(F_1, A) \cap (F_2, B)$”, is defined by;
$$(F_1, A) \cap (F_2, B) = (F_2, A \cap B),$$
where $F_2(\alpha) = F_1(\alpha) \cap F_2(\alpha) \forall \alpha \in A \cap B.$

Definition 3.10. If $(F_1, A)$ and $(F_2, B)$ are two soft normal mgroups over a mgroup $M$, then the union of $(F_1, A)$ and $(F_2, B)$, denoted by $(F_1, A) \cup (F_2, B) = (H, A \cup B)$, is defined by;
$$H(\alpha) = \begin{cases} F(\alpha) & \text{if } \alpha \in (A - B); \\ G(\alpha) & \text{if } \alpha \in (B - A); \\ F(\alpha) \cup G(\alpha) & \text{if } \alpha \in (A \cap B). \end{cases}$$

Proposition 3.11. Let $M$ be a mgroup over a group $X$ and $(F_1, A)$ and $(F_2, B)$ be two soft normal mgroups over $M$. Then their intersection $(F_1, A) \cap (F_2, B)$ is a soft normal mgroup over $M$.

Proof. Let $(F_1, A) \cap (F_2, B) = (H, A \cap B)$ such that
$$H(\alpha) = F_1(\alpha) \cap F_2(\alpha) \text{ for all } \alpha \in (A \cap B)$$
But $F_1(\alpha) \cap F_2(\alpha) \in Attr(M)$ (see proposition 3.2(i))
Hence, $H(\alpha) \in Attr(M)$
In particular, $(F_1, A) \cap (F_2, B)$ is a soft normal mgroup.

Proposition 3.12. Let $M$ be a mgroup over a group $X$ and $(F_1, A)$ and $(F_2, B)$ be two soft normal mgroups over $M$. Then their union $(F_1, A) \cup (F_2, B)$ is a soft normal mgroup over $M$.

Proof. Let $(F_1, A) \cup (F_2, B) = (H, A \cup B)$ such that
$$H(\alpha) = \begin{cases} F_1(\alpha) & \text{if } \alpha \in (A - B); \\ F_2(\alpha) & \text{if } \alpha \in (B - A); \\ F_1(\alpha) \cup F_2(\alpha) & \text{if } \alpha \in (A \cap B). \end{cases}$$
Clearly, $F_1(\alpha), F_2(\alpha)$ and $F_1(\alpha) \cup F_2(\alpha)$ are normal submgroups (by definition and proposition 3.2 (ii)).
Thus, $(F_1, A) \cup (F_2, B)$ is a soft normal mgroup.

Proposition 3.13. If $(F_1, A)$ and $(F_2, B)$ are two soft normal mgroups over $M$, then $(F_1, A) \land (F_2, B)$ is a soft normal mgroup over $M$.

Proof.
Recall, \((F_1, A) \text{ AND } (F_2, B)\) denoted by \((F_1, A) \wedge (F_2, B)\) is defined by \((F_1, A) \wedge (F_2, B) = (F_1, A \times B)\).

Such that \(H(\alpha_1, \alpha_2) = F_1(\alpha_1) \cap F_2(\alpha_2), \forall (\alpha_1, \alpha_2) \in A \times B\).

Since \((F_1, A)\) and \((F_2, B)\) are soft normal mgroups over \(M\), then the submsets \(F_1(\alpha_1)\) and \(F_2(\alpha_2)\) are both normal submgroups of \(M\) (by definition).

In particular, \(F_1(\alpha_1) \cap F_2(\alpha_2)\) is a normal submgroup (see proposition 3.2(i)).

Thus, \((F_1, A) \wedge (F_2, B)\) is a soft normal mgroup.

**Proposition 3.14.** If \((F_1, A)\) and \((F_2, B)\) are two soft normal mgroups over \(M\), then \((F_1, A) \vee (F_2, B)\) is a soft normal mgroup over \(M\).

**Proof.**

Recall, \((F_1, A) \text{ OR } (F_2, B)\) denoted by \((F_1, A) \lor (F_2, B)\) is defined by \((F_1, A) \lor (F_2, B) = (F_1, A \times B)\),

where \(H(\alpha_1, \alpha_2) = F_1(\alpha_1) \cup F_2(\alpha_2), \forall (\alpha_1, \alpha_2) \in A \times B\).

Since \((F_1, A)\) and \((F_2, B)\) are soft normal mgroups over \(M\), then the submsets \(F_1(\alpha_1)\) and \(F_2(\alpha_2)\) are both normal submgroups of \(M\) (by definition) and 

\(F_1(\alpha_1) \cup F_2(\alpha_2)\) is a normal submgroup (see proposition 4.2.2(ii)).

Therefore, \(H(\alpha_1, \alpha_2)\) is normal submgroup of \(M, \forall (\alpha_1, \alpha_2) \in A \times B\).

In particular, \((F_1, A) \lor (F_2, B)\) is a soft normal mgroup.

**Definition 3.15.** Let \((F_1, A)\) and \((F_2, B)\) be a soft normal mgroups over \(M \in MG(X)\). Then we say that \((F_2, B)\) is sub soft normal mgroup of \((F_1, A)\), written as \((F_2, B) \trianglelefteq (F_1, A)\), if \(F_2(\alpha)\) is a normal submgroup of \(F_1(\alpha)\), \(\forall \alpha \in B\).

**Proposition 3.16.** If \((F_1, A)\), \((F_2, B)\) and \((F_3, C)\) are soft normal mgroups over \(M \in MG(X)\) such that \((F_1, A) \trianglelefteq (F_2, B) \trianglelefteq (F_3, C)\) then \((F_1, A) \trianglelefteq (F_3, C)\).

**Proof.**

\((F_1, A) \trianglelefteq (F_2, B) \implies F_1(\alpha) \subseteq F_2(\alpha) \text{ and } A \subseteq B, \forall \alpha \in A \) (by definition 4.3.12) \hfill (1)

\((F_2, B) \trianglelefteq (F_3, C) \implies F_2(\alpha) \subseteq F_3(\alpha) \text{ and } B \subseteq C, \forall \alpha \in A \) (by definition 4.3.12) \hfill (2)

But \(A \subseteq B\) and \(B \subseteq C\) \implies \(A \subseteq C\) (\(\subseteq\) is a transitive) \hfill (3)

Using (1), (2) and (3) we have;

\(F_1(\alpha) \subseteq F_2(\alpha) \subseteq F_3(\alpha)\) i.e. \(F_1(\alpha) \subseteq F_2(\alpha) \subseteq F_3(\alpha), \forall \alpha \in A \subseteq C\)

Since \(F_2(\alpha)\) and \(F_3(\alpha)\) are normal submgroups.

Thus, \((F_1, A) \trianglelefteq (F_3, C)\).

**IV. CONCLUSIONS**

We have developed the concepts of normal submultigroups and soft normal multigroups with some of their algebraic structures such as intersection, union of normal submultigroups is also normal submultigroup of a given multigroup, the inverse of any normal submultigroup of a multigroup is a normal submultigroup and for any normal submultigroup, the root (support) set is a normal subgroup of the underlying group. We also showed that under the isomorphism function between any two groups, the image of a normal submultigroup under the isomorphism is a...
normal submultigroup and the inverse image of a normal submultigroup under the isomorphism is a normal submultigroup. Finally, we defined operations on soft normal multigroups such as intersection, union, AND, OR operations and discovered that such operations are closed under soft normal multigroups.

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