Periodic Positive Solutions of a Discrete Food Chain Predator-Prey Model

Mohammed El Amine Boubekeur
Biomathematics Laboratory,
Univ. Sidi Bel Abbes, BP89, 22000 Algeria
bbramine@gmail.com

Ahmed Lakmeche
Biomathematics Laboratory,
Univ. Sidi Bel Abbes, BP89, 22000 Algeria
lakahmed2000@yahoo.fr

Abdelkader Lakmeche
Biomathematics Laboratory,
Univ. Sidi Bel Abbes, BP89, 22000 Algeria
lakmeche@yahoo.fr

Abstract-In this work, we investigate a discrete mathematical model describing the evolution of two populations in interaction. The populations are described by the evolution of three level predator-prey model where the predator feed on both juvenile and adult preys in different ways. First we prove the existence of periodic positive solution. After that we give numerical simulations to illustrate our results.

I. INTRODUCTION

In the last five decades mathematical ecology was investigated by many researchers, see [10] and [13], and the references cited therein. The stage structured population has drawn the attention of many scholars, see [1], [5], [7] and [12], and the references cited therein.

Peter and Christopher [1] have considered the stability analysis of the following stage structured density-independent prey growth where the juvenile prey population is vulnerable

\[
\begin{align*}
\frac{dN_1}{dt} &= B_2N_2 - d_1N_1 - g(1 - \alpha_3N_1)N_1 - \frac{p_1N_1P}{1 + p_1hN_1} \\
\frac{dN_2}{dt} &= GN_1 - d_2N_2 \\
\frac{dP}{dt} &= P \left[ \frac{ep_1N_1}{1 + p_1hN_1} - D \right]
\end{align*}
\]

where \(N_1\) and \(N_2\) denote the population densities of juvenile and adult prey and \(P\) is that of the predator. The parameters \(d_1\), \(d_2\) and \(D\) are the per capita death rates of juvenile prey, adult prey and predator respectively. Here the predation is limited to juvenile prey with \(p_1\) as per capita capture rate of juveniles by the predator, \(h\) is the corresponding handling time which includes the time spent for pursuing, capturing, killing, eating as well as digestion of each captured prey item. The handling time interval starts once the prey has been spotted, \(e\) is the conversion efficiency of ingested juvenile prey into new predator individuals (see [1]). The authors found out that an alternative equilibrium can emerge if the
density-dependence on juvenile’s growth rate and adult’s birth rate is assumed, they assumed also that competition occurs in both prey stages for the following model

\[
\begin{align*}
\frac{dN_1}{dt} &= B_2(1 - \alpha_2N_2)N_2 - d_1N_1 - G(1 - \alpha_1N_1)N_1 - \frac{p_1N_1P}{1 + p_1hN_1}, \\
\frac{dN_2}{dt} &= G(1 - \alpha_1N_1)N_1 - d_2N_2, \\
\frac{dP}{dt} &= P \left[ \frac{ep_1N_1}{1 + p_1hN_1} + D \right].
\end{align*}
\]  

where \( \alpha_1 \) and \( \alpha_2 \) are the competition coefficients in both prey population stages.

Later in \cite{7}, the authors investigated the following model

\[
\begin{align*}
\frac{dN_1}{dt} &= B_2(1 - \alpha_2N_2)N_2 - d_1N_1 - G(1 - \alpha_1N_1)N_1 - \frac{p_1N_1P}{1 + p_1hN_1}, \\
\frac{dN_2}{dt} &= G(1 - \alpha_1N_1)N_1 - d_2N_2, \\
\frac{dP}{dt} &= P \left[ \frac{p_1N_1}{1 + p_1hN_1} + D \right].
\end{align*}
\]  

and

\[
\begin{align*}
\frac{dN_1}{dt} &= B_2(1 - \alpha_2N_2)N_2 - d_1N_1 - G(1 - \alpha_1N_1)N_1 - \frac{p_1N_1P}{1 + p_1hN_1}, \\
\frac{dN_2}{dt} &= G(1 - \alpha_1N_1)N_1 - d_2N_2 - \frac{p_2N_2P}{1 + p_2hN_2}, \\
\frac{dP}{dt} &= P \left[ \frac{p_2N_2}{1 + p_2hN_2} + D \right].
\end{align*}
\]

They obtained sufficient conditions under which the existence a stable limit cycle. As their model incorporates non-periodic parameters.

Cushing \cite{3} has already pointed out that nature and environment require some times the parameters to be periodic. One of the fundamental question in mathematical ecology is the existence of positive periodic solution i.e the existence of cycle.

Other interested works on discrete population models for prey-predator can be found in the following papers \cite{2}, \cite{6}, \cite{8} and \cite{9}.

Motivated by the work in \cite{7}, in this work we will discuss the existence of periodic positive solution of the following discrete time system where the predation is limited to both prey stages

\[
\begin{align*}
N_1(n+1) &= N_1(n)\exp \left \{ B_2(n)(1 - \alpha_2(n)N_2(n))\frac{N_2(n)}{N_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)N_1(n)) - \frac{p_1(n)P(n)}{1 + p_1(n)h_1N_1(n)} \right \}, \\
N_2(n+1) &= N_2(n)\exp \left \{ G(n)(1 - \alpha_1(n)N_1(n))\frac{N_1(n)}{N_2(n)} - d_2(n) - \frac{p_2(n)P(n)}{1 + p_2(n)h_2N_2(n)} \right \}, \\
P(n+1) &= P(n)\exp \left \{ - \frac{p_1(n)N_1(n)}{1 + p_1(n)h_1N_1(n)} + \frac{p_2(n)N_2(n)}{1 + p_2(n)h_2N_2(n)} - D(n) \right \}.
\end{align*}
\]

Our paper is organized as follow, in the next section we give conditions to obtain the existence of positive periodic solution, in section three we give numerical simulations, the last section contains conclusions on this work.
II. Existence of Positive Periodic Solution

Let $X, Z$ be Banach spaces, $\Omega \subset X$ a bounded open set with closure $\bar{\Omega}$, and

$$L : \text{dom}L \subset X \to Z.$$  

L will be calling Fredholm mapping with index zero if (i)

(i) $L$ is linear and $\text{Im}L$ is closed in $Z$.
(ii) ker$L$ and coker$L$ have a finite dimension and

$$\dim \text{ker}L = \dim \text{coker}L = \dim \text{im}L$$

If $L$ is a Fredholm mapping with index 0 and there exist continuous projectors

$$P : X \to X \quad \text{and} \quad Q : Z \to Z,$$

such that $\text{im}P = \text{ker}L, \text{im}L = \text{ker}Q = \text{im}(I - Q)$, it follows that $L|_{\text{dom}L \cap \ker P} : (I - P)X \to \text{im}L$ is invertible. We denote its inverse by $K_P$.

If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\bar{\Omega}$ if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact, i.e. it is continuous and $K_P(I - Q)N(\bar{\Omega})$ is relatively compact, where $K_P : \text{im}L \to \text{dom}L \cap \ker P$ is the inverse of the restriction $LP$ of $L$ to $\text{dom}L \cap \ker P$, so that $LK_P = I$ and $KP = I - P$.

Since $Q$ is isomorphic to $\ker L$, then there exist an isomorphic $J : \text{im}Q \to \ker L$.

Lemma 2.1 (i). (Continuation theorem) Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $\bar{\Omega}$. Suppose

(a) for each $\lambda \in (0, 1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$;
(b) $QN \neq 0$ for each $x \in \partial \Omega \cap \ker L$ and $\deg \{ JQN, \Omega \cap \ker L, 0 \} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution lying in $\text{dom}L \cap \bar{\Omega}$.

We will use the following notations

$$I_\omega = \{0, 1, 2, \ldots, \omega - 1\}, \quad \bar{f} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} f(n)r$$

$$a^I = \min_{n \in I_\omega} a(n) \quad a'' = \max_{n \in I_\omega} a(n)$$

where $\{f(n)\}$ is an $\omega$-periodic sequence of real numbers defined for $s \in \mathbb{Z}$. 


Lemma 2.2 (I1). Let $f : \mathbb{Z} \to \mathbb{R}$ be $\omega$-periodic, i.e., $f(n + \omega) = f(n)$. Then for any fixed $n_1, n_2 \in I_{\omega}$, and any $n \in \mathbb{Z}$, one has

$$f(n) \geq f(n_1) + \sum_{n=0}^{\omega-1} |f(n+1) - f(n)|$$

and

$$f(n) \leq f(n_2) - \sum_{n=0}^{\omega-1} |f(n+1) - f(n)|.$$  

Theorem 2.1. Assume that the following conditions are satisfied

(a) $\max \{\bar{R}, \bar{S}\} < \bar{D}$, where $\bar{R} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_1(n) e^{M_1}}{1 + p_1(n) h_1 e^{M_1}}$ and $\bar{S} = \frac{1}{\omega} \sum_{n=0}^{\omega-1} \frac{p_2(n) e^{M_2}}{1 + p_2(n) h_2 e^{M_2}}$.

(b) $\bar{\alpha}_2 < \frac{\beta_2 e^{M_1} + 2 \beta_2}{2 \alpha_3}$. 

(c) $\deg \{JQNu, \Omega \cap \mathbb{R}^3, 0\} \neq 0$. 

Then system (5) has at least one $\omega$-periodic solution.

Proof. Let $I_3 := \{u = \{u(n)\} : u(n) \in \mathbb{R}^3, n \in \mathbb{Z}\}$. 

For $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, define $|a| := \max \{a_1, a_2, a_3\}$. Let $I_\omega (\subset I_3)$ denotes the subset of all $\omega$-periodic sequences equipped with usual supremum norm $\|\cdot\|$, i.e.

$$\|u\| = \max_{n \in I_\omega} |u(n)| \quad \text{for} \quad u = \{u(n) : n \in \mathbb{Z}\} \in I_\omega.$$ 

The space $I_\omega$ is a finite dimensional Banach space.

Let

$$I_\omega^0 := \left\{ u = u(n) \in I_\omega : \sum_{n=0}^{\omega-1} u(n) = 0 \right\},$$

and

$$I_\omega^c := \left\{ u = u(n) \in I_\omega : u(n) = h \in \mathbb{R}^3, n \in \mathbb{Z} \right\}.$$ 


Then $I_\omega^0$ and $I_\omega^c$ are both closed linear subspaces of $I_\omega$. Moreover, $I_\omega = I_\omega^0 \oplus I_\omega^c$ and $\dim I_\omega^0 = 3$. Put $N_1(n) = \exp \{u_1(n)\}$, $N_2(n) = \exp \{u_2(n)\}$ and $P(n) = \exp \{u_3(n)\}$.

The system (5) can be reformulated as follows:

$$\Delta u_1(n) = u_1(s + 1) - u_1(n) = B_2(n) (1 - \alpha_2(n) \exp \{u_2(n)\}) \exp \{u_2(n) - u_1(n)\}$$

$$- d_1(n) - G(n)(1 - \alpha_1(n) \exp \{u_1(n)\}) - \frac{p_1(n) \exp \{u_3(n)\}}{1 + p_1(n) h_1 \exp \{u_1(n)\}},$$

$$\Delta u_2(n) = u_2(s + 1) - u_2(n) = G(n)(1 - \alpha_1(n) \exp \{u_1(n)\}) \exp \{u_1(n) - u_2(n)\} - \frac{p_2(n) \exp \{u_3(n)\}}{1 + p_2(n) h_2 \exp \{u_2(n)\}} - d_2(n),$$

$$\Delta u_3(n) = u_3(s + 1) - u_3(n) = -D(n) + \frac{p_1(n) \exp \{u_1(n)\}}{1 + p_1(n) h_1 \exp \{u_1(n)\}} + \frac{p_2(n) \exp \{u_2(n)\}}{1 + p_2(n) h_2 \exp \{u_2(n)\}}.$$
In order to implant our problem into framework of continuation theorem, let us first define

\[ I^\omega = X = Z = \{ u(n) = (u_1(n), u_2(n), u_3(n))^T \in \mathbb{R}^3, u(n + \omega) = u(n) \} \]

and

\[ \|u\| = \|(u_1(n), u_2(n), u_3(n))^T\| = \max_{n \in I_\omega} |u_1(n)| + \max_{n \in I_\omega} |u_2(n)| + \max_{n \in I_\omega} |u_3(n)| \]

for any \( u \in X \) (or \( Z \)). Then \( X \) and \( Z \) are Banach spaces with the norm \( \|\cdot\| \).

Let

\[
N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \end{bmatrix} = \begin{pmatrix} B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n)} - d_1(n) - G(n)(1 - \alpha_1(n)\exp\{u_1(n)\}) - \frac{p_1(n)\exp\{u_3(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} \\ G(n)(1 - \alpha_1(n)\exp\{u_1(n)\})\exp\{u_1(n) - u_2(n)\} - \frac{p_2(n)\exp\{u_3(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} - d_2(n) \\ -D(n) + \frac{p_1(n)\exp\{u_1(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} + \frac{p_2(n)\exp\{u_2(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \\ \end{pmatrix}
\]

and \( Lu = \Delta u(n) \), where \( \Delta u(n) = (\Delta u_1(n), \Delta u_2(n), \Delta u_3(n))^T \).

Then from Lemma 2.1 in [11], we have \( \ker L = \mathbb{R}^2 \), \( \text{Im} L = \{ z \in \mathbb{Z} : \sum_{n=0}^{\omega-1} z(n) = 0 \} \) is closed in \( Z \) and \( \dim \ker L = 3 = \text{codim} \text{Im} L \).

Therefore, \( L \) is a Fredholm mapping of index zero. Hence, there exist two continuous projectors \( P : X \to X \) and \( Q : Z \to Z \) such that \( P x = \frac{1}{\omega} \sum_{n=0}^{\omega-1} x(n), x \in X \), \( Q z = \frac{1}{\omega} \sum_{n=0}^{\omega-1} z(n), z \in Z \), \( \text{Im} P = \ker L \) and \( \text{Im} L = \ker Q = \text{Im}(I - Q) \).

Furthermore, the generalized inverse of \( L \), \( K_p : L \to \ker P \cap \text{dom} L \), is given by

\[
K_p(z) = \sum_{n=0}^{\omega-1} z(n) - \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - s)z(n).
\]

Then \( QN : X \to Z \) and \( K_p(I - Q) : \text{Im} L \to \ker P \cap \text{dom} L \), read

\[
QNw = \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta u(n), \frac{1}{\omega} \sum_{n=0}^{\omega-1} \Delta v(n) \right)^T.
\]

That is

\[
QNw = \left( \frac{1}{\omega} \sum_{n=0}^{\omega-1} \left( B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n)} - d_1(n) - G(n)(1 - \alpha_1(n)\exp\{u_1(n)\}) - \frac{p_1(n)\exp\{u_3(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} \right) \\ G(n)(1 - \alpha_1(n)\exp\{u_1(n)\})\exp\{u_1(n) - u_2(n)\} - \frac{p_2(n)\exp\{u_3(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} - d_2(n) \\ -D(n) + \frac{p_1(n)\exp\{u_1(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} + \frac{p_2(n)\exp\{u_2(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right)
\]

and

\[
K_p(I - Q)Nu = (\Phi_1, \Phi_2, \Phi_3)^T
\]

where

\[
\Phi(u(n)) = \sum_{n=0}^{\omega-1} \Delta u(n) - \frac{1}{\omega} \sum_{n=0}^{\omega-1} (\omega - s)\Delta u(n) - \left( \frac{n}{\omega} - \frac{\omega+1}{2\omega} \right) \sum_{n=0}^{\omega-1} \Delta u(n).
\]
Now we reach the position to search for an appropriate open bounded subset \( \Omega \). Suppose that the Ascoli theorem, we see that \( X \) is a finite dimensional Banach space, using the Ascoli-Ascoli theorem, and they are mapping bounded continuous functions to bounded continuous functions. Since \( X \) is a finite dimensional Banach space, the Ascoli-Ascoli theorem, we see that \( QN(\bar{\Omega}) \) and \( K_p(I - Q)N(\bar{\Omega}) \) are relatively compact for any open bounded set \( \Omega \subset \mathbb{X} \). Moreover, \( QN(\bar{\Omega}) \) is bounded. Then \( N \) is \( L \)-compact on \( \bar{\Omega} \) for any open bounded \( \Omega \subset \mathbb{X} \).

Now we reach the position to search for an appropriate open bounded subset \( \Omega \) for the application of the continuation theorem, corresponding to the operator equation \( Lu = \lambda Nu, \lambda \in (0, 1) \).

We have

\[
\Delta u_1(n) = \lambda \left\{ B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n) - u_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)e^{u_1(n)}) - \frac{p_1(n)e^{u_1(n)}}{1 + p_1(n)h_1e^{u_1(n)}} \right\},
\]

\[
\Delta u_2(n) = \lambda \left\{ G(n)(1 - \alpha_1(n)\exp u_1(n))\exp u_1(n) - u_2(n) - \frac{p_2(n)\exp u_2(n)}{1 + p_2(n)h_2\exp u_2(n)} - d_2(n) \right\}.
\]

\[
\Delta u_3(n) = \lambda \left\{ -D(n) + \frac{p_1(n)\exp u_1(n)}{1 + p_1(n)h_1\exp u_1(n)} + \frac{p_2(n)\exp u_2(n)}{1 + p_2(n)h_2\exp u_2(n)} \right\}.
\]

Suppose that \( u = u(n) \in \mathbb{X} \) is an arbitrary solution of system (6)-(8) for certain \( \lambda \in (0, 1) \).

Summing (6)-(8) from 0 to \( \omega - 1 \), we obtain

\[
\sum_{n=0}^{\omega-1} \left\{ B_2(n)(1 - \alpha_2(n)e^{u_2(n)})e^{u_2(n) - u_1(n)} - d_1(n) - G(n)(1 - \alpha_1(n)e^{u_1(n)}) - \frac{p_1(n)e^{u_1(n)}}{1 + p_1(n)h_1e^{u_1(n)}} \right\} = 0,
\]

\[
\sum_{n=0}^{\omega-1} \left\{ G(n)(1 - \alpha_1(n)\exp u_1(n))\exp u_1(n) - u_2(n) - \frac{p_2(n)\exp u_2(n)}{1 + p_2(n)h_2\exp u_2(n)} - d_2(n) \right\} = 0,
\]

\[
\sum_{n=0}^{\omega-1} \left\{ -D(n) + \frac{p_1(n)\exp u_1(n)}{1 + p_1(n)h_1\exp u_1(n)} + \frac{p_2(n)\exp u_2(n)}{1 + p_2(n)h_2\exp u_2(n)} \right\} = 0.
\]
That is
\[
\sum_{n=0}^{\omega-1} \left\{ B_2(n)(1 - \alpha_2(n)e^{\omega_2(n)})e^{\omega_2(n)} - u_1(n) - G(n)(1 - \alpha_1(n)e^{\omega_1(n)}) - \frac{p_1(n)e^{\omega_1(n)}}{1 + p_1(n)h_1e^{\omega_1(n)}} \right\} = \sum_{n=0}^{\omega-1} d_1(n) = \tilde{d}_1\omega, \tag{9}
\]
\[
\sum_{n=0}^{\omega-1} \left\{ G(n)(1 - \alpha_1(n)\exp\{u_1(n)\})\exp\{u_1(n) - u_2(n)\} - \frac{p_2(n)\exp\{u_3(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right\} = \tilde{d}_2\omega, \tag{10}
\]
\[
\sum_{n=0}^{\omega-1} \left\{ \frac{p_1(n)\exp\{u_1(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} + \frac{p_2(n)\exp\{u_2(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right\} = \sum_{n=0}^{\omega-1} D(n) = \tilde{D}\omega. \tag{11}
\]

From (6), (11), it follows that
\[
\sum_{n=0}^{\omega-1} |u_1(n + 1) - u_1(n)| = \lambda \sum_{n=0}^{\omega-1} \left| B_2(n)(1 - \alpha_2(n)e^{\omega_2(n)})e^{\omega_2(n)} - u_1(n) - G(n)(1 - \alpha_1(n)e^{\omega_1(n)}) - \frac{p_1(n)e^{\omega_1(n)}}{1 + p_1(n)h_1e^{\omega_1(n)}} \right| < \sum_{n=0}^{\omega-1} d_1(n) + \sum_{n=0}^{\omega-1} \left| B_2(n)(1 - \alpha_2(n)e^{\omega_2(n)})e^{\omega_2(n)} - u_1(n) - G(n)(1 - \alpha_1(n)e^{\omega_1(n)}) - \frac{p_1(n)e^{\omega_1(n)}}{1 + p_1(n)h_1e^{\omega_1(n)}} \right| = (2\tilde{d}_1)\omega, \tag{12}
\]
\[
\sum_{n=0}^{\omega-1} |u_2(n + 1) - u_2(n)| = \lambda \sum_{n=0}^{\omega-1} \left| G(n)(1 - \alpha_1(n)\exp\{u_1(n)\})\exp\{u_1(n) - u_2(n)\} - \frac{p_2(n)\exp\{u_3(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right| < \sum_{n=0}^{\omega-1} d_2(n) + \sum_{n=0}^{\omega-1} \left| G(n)(1 - \alpha_1(n)\exp\{u_1(n)\})\exp\{u_1(n) - u_2(n)\} - \frac{p_2(n)\exp\{u_3(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right| = (2\tilde{d}_2)\omega, \tag{13}
\]
\[
\sum_{n=0}^{\omega-1} |u_1(n + 1) - u_1(n)| = \lambda \sum_{n=0}^{\omega-1} \left| -D(n) + \frac{p_1(n)\exp\{u_1(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} + \frac{p_2(n)\exp\{u_2(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right| < \sum_{n=0}^{\omega-1} D(n) + \sum_{n=0}^{\omega-1} \left( \frac{p_1(n)\exp\{u_1(n)\}}{1 + p_1(n)h_1\exp\{u_1(n)\}} + \frac{p_2(n)\exp\{u_2(n)\}}{1 + p_2(n)h_2\exp\{u_2(n)\}} \right) = (2\tilde{D})\omega, \tag{14}
\]
That is
\[
\sum_{n=0}^{\omega-1} |u_1(n + 1) - u_1(n)| < (2\tilde{d}_1)\omega, \tag{13}
\]
\[
\sum_{n=0}^{\omega-1} |u_2(n + 1) - u_2(n)| < (2\tilde{d}_2)\omega, \tag{14}
\]
\[
\sum_{n=0}^{\omega-1} |u_3(n + 1) - u_3(n)| < (2\tilde{D})\omega. \tag{15}
\]
Since \(u(n) = (u_1(n), u_2(n), u_3(n))^T \in \mathbb{R}_+^3\), there exist \(\zeta_i, \eta_i \in I_\omega\), such that
\[
u_1(\eta_i) = \min_{n \in I_\omega} u_1(n), \quad u_i(\zeta_i) = \max_{n \in I_\omega} u_i(n), \quad i = 1, 2, 3. \tag{16}
\]
From (9) and (16), we see that
\[
u_1(\eta_1) \leq \ln \left( \frac{\tilde{d}_1 + \tilde{G}}{\tilde{G}_1} \right). \tag{17}
\]
From Lemma 2.2, we obtain
\[
u_1(n) \leq \nu_1(\eta_1) + \sum_{n=0}^{\omega-1} |u_1(n + 1) - u_1(n)| \leq \ln \left( \frac{\tilde{d}_1 + \tilde{G}}{\tilde{G}_1} \right) + (2\tilde{d}_1)\omega = M_1. \tag{18}
\]
From (10), (11) and (16), we obtain
\[ u_2(\eta_2) \leq \ln \left( \frac{\bar{d}_1 + \bar{G}}{\bar{a}_1 \bar{d}_2} \right) + (2\bar{d}_1)\omega. \] (19)

Using lemma 2.2 and (19), we get
\[ u_2(n) \leq u_2(\eta_2) + \sum_{n=0}^{\omega-1} |u_2(n+1) - u_2(n)| \leq \ln \left( \frac{\bar{d}_1 + \bar{G}}{\bar{a}_1 \bar{d}_2} \right) + 2(\bar{d}_1 + \bar{d}_2)\omega = M_2. \] (20)

From (11) and (16), we see that
\[ u_1(\zeta_1) \geq \ln \left( \frac{\bar{D}_2 \omega - \sum_{n=0}^{\omega-1} \frac{p_2(n)e^{M_2}}{1 + p_2(n)h_2 e^{M_2}}}{\bar{p}_2 \omega} \right). \] (21)

Using lemma 2.2 and (23), we get
\[ u_1(n) \geq u_1(\zeta_1) - \sum_{n=0}^{\omega-1} |u_1(n+1) - u_1(n)| \geq \ln \left( \frac{\bar{D}_2 \omega - \sum_{n=0}^{\omega-1} \frac{p_2(n)e^{M_2}}{1 + p_2(n)h_2 e^{M_2}}}{\bar{p}_2 \omega} \right) - (2\bar{d}_1)\omega = M_3. \] (22)

From (11), (22) and (16), we see that
\[ u_2(\zeta_2) \leq \begin{cases} \ln \left( \frac{\bar{d}_1}{\bar{B}_2} \right) + M_3, \\ \ln \left( \frac{\bar{D}_2 \omega - \sum_{n=0}^{\omega-1} \frac{p_1(n)e^{M_1}}{1 + p_1(n)h_1 e^{M_1}}}{\bar{p}_2 \omega} \right). \end{cases} \] (23)

Using lemma 2.2 and (23), we get
\[ u_2(n) \leq \begin{cases} \ln \left( \frac{\bar{d}_1}{\bar{B}_2} \right) + M_3, \\ \ln \left( \frac{\bar{D}_2 \omega - \sum_{n=0}^{\omega-1} \frac{p_1(n)e^{M_1}}{1 + p_1(n)h_1 e^{M_1}}}{\bar{p}_2 \omega} \right) - 2\omega\bar{d}_2 = M_4. \end{cases} \] (24)

From (8) one can obtain
\[ |u_3(n)| \leq |u_3(0)| + \bar{\lambda} \left| \sum_{k=0}^{\omega-2} f(u_1(k), u_2(k)) \right| < |u_3(0)| + (\omega - 1)f^u. \] (25)

Thus equation (25) leads to
\[ u_3(\zeta_3) > - |u_3(0)| - (\omega - 1)f^u, \] (26)
\[ u_3(\eta_3) < |u_3(0)| + (\omega - 1)f^u. \] (27)

It is easy to obtain the upper bound of \( u_3(0) \) from either (9) or (10), in contrast the lower bound is obtained using both (9) and (11). From Lemma 2.2 and (26), we obtain
\[ n_3(n) \geq u_3(\xi) - \sum_{n=0}^{2^\omega-2} |u_3(n+1) - u_3(n)| \geq -|u_3(0)| - f''(\omega - 1) - (2\tilde{D})\omega = M_5, \tag{28} \]
\[ n_3(n) \leq u_3(\eta) + \sum_{n=0}^{2^\omega-2} |u_3(n+1) - u_3(n)| \leq |u_3(0)| + f''(\omega - 1) + (2\tilde{D})\omega = M_6, \tag{29} \]

where \( f'' = e^{M_1} + e^{M_2} \).

From (18) and (22) we have
\[ \max_{n \in I_{\omega}} |u_1(n)| \leq \max \{|M_1|, |M_3|\} =: \bar{M}_1. \]

From (20) and (24) we have
\[ \max_{n \in I_{\omega}} |u_2(n)| \leq \max \{|M_2|, |M_4|\} =: \bar{M}_2. \]

Thus from (28) we have
\[ \max_{n \in I_{\omega}} |u_3(n)| \leq \max \{|M_5|, |M_6|\} =: \bar{M}_3. \]

We have \( M_i, \bar{M}_j (i = 1, 2, 3, 4, 5, 6) \) and \( j = 1, 2, 3 \) are independent of \( \lambda \). Denote \( M = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_4 \), where \( \bar{M}_4 \) is taken sufficiently large such that the unique solution of system (6)-(8) satisfies \( \|(u_1, u_2, u_3)^T\| < M \).

Let
\[ \Omega = \{u(n) = (u_1(n), u_2(n), u_3(n))^T \in X : \|u\| < M\}. \]

Clearly, \( \Omega \) satisfies condition (a) of Lemma 2.1. If \( u \in \partial \Omega \cap \ker L \), then \( u \) is a constant with \( \|u\| = M \).

Hence, we have
\[
\begin{pmatrix}
    g_1(u_1, u_2, u_3) \\
    g_2(u_1, u_2, u_3) \\
    g_3(u_1, u_2, u_3)
\end{pmatrix} =
\begin{pmatrix}
    B_2(1-\alpha_2 e^{\theta_1}) e^{\theta_2 - \alpha_1} - \tilde{a}_1 - G(1-\alpha_1 e^{\theta_1}) - \frac{1}{\omega} \sum_{n=0}^{2^\omega-1} \frac{p_1(n) e^{\theta_1}}{1 + p_1(n) h_1 e^{\theta_1}} \\
    -D + \frac{1}{\omega} \sum_{n=0}^{2^\omega-1} \frac{p_1(n) e^{\theta_1}}{1 + p_1(n) h_1 e^{\theta_1}} \\
    0
\end{pmatrix} = \begin{pmatrix}
    0 \\
    0 \\
    0
\end{pmatrix}.
\]

Taking \( J = \text{Im}Q \rightarrow \ker L, (u, v)^T \rightarrow (u, v)^T \). We have
\[
\deg \{JQNu, \Omega \cap \ker L, 0\} = \deg \{JQNu, \Omega \cap \mathbb{R}^3, 0\} = \text{sign} \begin{pmatrix}
    \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \frac{\partial g_1}{\partial u_3} \\
    \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \frac{\partial g_2}{\partial u_3} \\
    \frac{\partial g_3}{\partial u_1} & \frac{\partial g_3}{\partial u_2} & \frac{\partial g_3}{\partial u_3}
\end{pmatrix}.
\]

In view of condition (c) of Theorem (2.1), we know that \( \deg \{JQNu, \Omega \cap \mathbb{R}^3, 0\} \neq 0 \).

**Corollary 2.1.** If \( p_1 \) is sufficiently small than \( p_2 \), and vice versa, then Theorem (2.1) holds true.

**Proof.** If \( p_1 << p_2 \), then
\[ ccl \text{deg}\{JQNu, \Omega \cap \ker L, 0\} \simeq \frac{1}{h^2} \left( -\tilde{B}_2(1 - \tilde{a}_2 e^{\mu_2}) e^{\mu_2} + \tilde{G} \tilde{a}_1 e^{\mu_1} \right) \times \left( \sum_{n=0}^{\infty} \frac{p_2(n) e^{\mu_2}}{1 + p_2(n) h_2 e^{\mu_2}} \right) \]

since \( JQN \) is continuous at 0, it follows that it is continuous at the neighborhood of 0, then \( \text{deg}\{JQNu, \Omega \cap \ker L, 0\} \neq 0. \)

The same proof is adopted for \( p_2 \ll p_1. \)

**Corollary 2.2.** If \( p_1 \) and \( p_2 \) are constants then Theorem \([2,1]\) holds true, with

\[ D > \max\{R,S\}, \text{ where } R = \frac{p_1 e^{M_1}}{1 + p_1 h_1 e^{M_1}} \text{ and } S = \frac{p_2 e^{M_2}}{1 + p_2 h_2 e^{M_2}}. \]

### III. NUMERICS AND SIMULATIONS

In this section, we use some computer simulations to show the feasibility our previous results. Let’s consider the following four periodic system

\[
\begin{aligned}
N_1(n+1) &= N_1(n) \exp \left\{ 1.81 + 0.01 \sin\left( \frac{\pi n}{2} \right) (1 - (1.3 + 0.02 \cos(\frac{\pi n}{2})) N_2(n) \right\} \frac{N_2(n)}{N_1(n)} \\
&\quad - (0.19 + 0.01 \sin(\frac{\pi n}{2})) - (0.8 + 0.02 \sin(\frac{\pi n}{2})) - (1 - (0.5 + 0.01 \cos(\frac{\pi n}{2})) N_2(n)) \\
&\quad - \left( \frac{\left( 1 + 0.01 \sin(\frac{\pi n}{2}) \right) P(n)}{1 + \left( 1 + 0.01 \sin(\frac{\pi n}{2}) \right) N_1(n)} \right),
\end{aligned}
\]

\[
\begin{aligned}
N_2(n+1) &= N_2(n) \exp \left\{ 0.8 + 0.02 \sin(\frac{\pi n}{2}) \left( 1 - (0.04 + 0.01 \cos(\frac{\pi n}{2})) N_1(n) \right) \frac{N_1(n)}{N_2(n)} \\
&\quad - \frac{\left( 0.7 + 0.01 \cos(\frac{\pi n}{2}) \right) P(n)}{1 + \left( 2 + 0.01 \sin(\frac{\pi n}{2}) \right) \left( 0.7 + 0.01 \cos(\frac{\pi n}{2}) \right) N_2(n)} - (0.31 + 0.01 \sin(\frac{\pi n}{2})) \right\},
\end{aligned}
\]

\[
\begin{aligned}
P(n+1) &= P(n) \exp \left\{ \frac{\left( 1 + 0.01 \sin(\frac{\pi n}{2}) \right) N_1(n)}{1 + \left( 1 + 0.01 \sin(\frac{\pi n}{2}) \right) N_1(n)} \\
&\quad + \frac{\left( 0.7 + 0.01 \cos(\frac{\pi n}{2}) \right) N_2(n)}{1 + \left( 2 + 0.01 \sin(\frac{\pi n}{2}) \right) \left( 0.7 + 0.01 \cos(\frac{\pi n}{2}) \right) N_2(n)} - (0.34 + 0.02 \cos(\frac{\pi n}{2})) \right\}.
\end{aligned}
\]
Figure 1. Juvenile prey curve for initial condition \( (N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80) \).

Figure 2. Adult prey curve for initial condition \( (N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80) \).

Figure 3. Predator curve for initial condition \( (N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80) \).

Figure 4. Predator curve (black line), adult prey curve (blue line) and juvenile curve (red line) for initial condition \( (N_1(0), N_2(0), P(0)) = (0.30, 0.64, 0.80) \).
IV. CONCLUDING REMARKS

In this work we have considered a discrete predator-prey mathematical model describing the interaction between predator and two levels of prey population constituted by juveniles and adults. Our aim is to obtain periodic positive solution of the model under study. We have obtained sufficient conditions to have positive periodic solution of the stated model, our results are illustrated by numerical simulations. In future work, we are planning to consider the case of delay and bifurcation analysis.

REFERENCES


