Modified Sirs Epidemic Model with Immigration and Saturated Incidence

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Abstract- The present mathematical model deals with the study of SIRS epidemic model with immigration and saturation type incidence. We start from formulation of model and analyze it. The disease free equilibrium and endemic equilibrium of the system are established. If $R_0 < 1$ the DFE (Disease Free Equilibrium) is globally stable and if $R_0 > 1$ then the endemic equilibrium is obtained which is globally stable. An example also provides to justify the stability.

Keywords: Mathematical Model, Equilibrium Point, Stability Analysis, Saturated incidence.

I. INTRODUCTION

Mathematical modeling is an important tool to understand and predict the spread of infectious diseases. The incidence is an epidemiological model is the rate at which the susceptible become infectious. Cappaso and Serio [10] introduced a saturated incidence rate into epidemic model. Mena – Lorca and Hethcote [2] also analyzed an SIRS model with the same saturation incidence. Porwal and Badshah studied a mathematical model with modified saturated incidence rate [4]. Several different incidence rates have been proposed by many researchers (see, Anderson and May [8], Elteva and Matias [3], Hethcote and Driessch [1], Porwal, et al. [4, 5, 6, 7]).

The emerging and re-emerging diseases have stimulated the interest in mathematical modeling. Models can provide estimates of underlying parameters of a real world problem which are difficult or expensive to obtain through experiment or otherwise. By estimating transmission rate, reproduction number and other variables and parameters, a model can predict whether the associated disease will spread through the population or dies out. It can also estimate the impact of a control measure and provide useful guidelines to public health for further efforts required for disease elimination.

In this paper, we investigate modified SIRS epidemic model with immigration and saturated type incidence, and modify the model of Pathak , et al. [9] by considering the immigration rate and disease induced death rate. We present modified mathematical model, analyze the model and obtain disease free and endemic equilibrium point and analyze for stability analysis. Further we also give an example for satisfaction of our results.
II. The Mathematical Model

2.1 Basic Model
Pathak, et al. [9] has proposed the following dynamical system of differential equations

\[
\begin{align*}
\frac{dS}{dt} &= b - ds - \frac{ksI}{1 + \alpha S + \beta I} + \gamma R + \delta \\
\frac{dI}{dt} &= \frac{ksI}{1 + \alpha S + \beta I} - (d + \mu)I \\
\frac{dR}{dt} &= \mu I - (d + \gamma)R
\end{align*}
\]

where \( S(t) \), \( I(t) \), \( R(t) \) represent the number of susceptible, infective and recovered individual at time \( t \) respectively. Here \( b \) is the recruitment rate of population, \( d \) is natural death rate of the population, \( k \) is the proportionality constant, \( \mu \) is the natural recovery rate of individual \( \gamma \) is the immunity loss rate constant and \( \alpha, \beta \) are the parameters which measures the effects of sociological, psychological mechanisms. The transmission rate \( \phi = \frac{kl}{1 + \alpha S + \beta I} \) displays a saturation effect of the population.

2.2 Model with Immigration
The model (2.1) with immigration and disease induced death is given by:

\[
\begin{align*}
\frac{dS}{dt} &= b - ds - \frac{ksI}{1 + \alpha S + \beta I} + \gamma R + \mu I + \delta \\
\frac{dI}{dt} &= \frac{ksI}{1 + \alpha S + \beta I} - (d + \mu + \theta)I \\
\frac{dR}{dt} &= \mu I - (d + \gamma)R
\end{align*}
\]

where \( \delta \) is the increase of susceptible at a constant rate and \( \theta \) is the disease induced death rate and other parameters have similar meanings as for as the model (2.1).

III Equilibrium of the System
To obtain equilibrium point, we make R.H.S. of all equation (2.2) equal to zero, i.e.

\[
\begin{align*}
b - dS - \frac{ksI}{1 + \alpha S + \beta I} + \gamma R + \mu I + \delta &= 0, \quad (3.1) \\
\frac{ksI}{1 + \alpha S + \beta I} - (d + \mu + \theta)I &= 0. \quad (3.2) \\
\mu I - (d + \gamma)R &= 0. \quad (3.3)
\end{align*}
\]

By equation (3.3)

\[
R = \frac{\mu I}{d + \gamma}. \quad (3.4)
\]

From equation (3.2)

\[
\frac{ksI}{1 + \alpha S + \beta I} - (d + \mu + \theta)I = 0, \text{ or } I = 0. \quad (3.5)
\]

If \( I = 0 \), then (3.4) gives \( R = 0 \), and then (3.1) gives

\[
S = \frac{b + \delta}{d}.
\]
Hence the disease free equilibrium of the system (2.2) is given by \( E_0 = E_0 \left( \frac{b+\delta}{d}, 0, 0 \right) \).

If \( I \neq 0 \) then by equation (3.5), we get
\[
S^* = \frac{(d + \mu + \theta)(1 + \beta I)}{k - \alpha(d + \mu + \theta)},
\]

From equation (3.1) and (3.2) we have
\[
b - dS - (d + \mu + \theta)I + \gamma R + \delta = 0,
\]

which employing (3.4) and (3.6), gives
\[
I^* = \frac{(d + \gamma)[(b + \delta)k - (b + \delta)\alpha(d + \mu + \theta) - d(d + \mu + \theta)]}{\beta k l(d + \mu + \theta)(d + \gamma) + (k - \alpha(d + \mu + \theta))[d(d + \mu + \gamma) + \theta(d + \gamma)]}
\]

Define the Basic Reproduction Number as follows:
\[
R_0 = \frac{(b + \delta)k - (b + \delta)\alpha(d + \mu + \theta)}{d(d + \mu + \theta)}
\]

Hence the endemic equilibrium of the system point (2.2) is \( E_1(S^*, I^*, R^*) \),

Where \( S^* \) is given by (3.6), \( I^* \) is given by (3.7) and \( R^* = R \) is given by (3.4).

IV Analysis of the Mathematical Model

**Theorem 1.** The plane \( S + I + R = \frac{b+\delta}{d} \) is a manifold of system (2.2) when \( \theta = 0 \).

**Proof:** Let \( S(t) + I(t) + R(t) = N(t) \).

Adding the three equations of system (2.2), we get
\[
\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt},
\]

\[
\frac{dN}{dt} = b - d(S + I + R) + \delta - \theta I
\]

\[
\frac{dN}{dt} = (b + \delta) - dN
\]

(for \( \theta = 0 \))

Obviously \( N(t) = \frac{b+\delta}{d} \) is the solution of (4.1).

For general solution of equation (4.1)
\[
\frac{dN}{(b+\delta) - dN} = dt,
\]

After integrating, we get
\[
\log\left(\frac{(b+\delta) - dN}{-d}\right) = t + C.
\]

at \( t = t_0 \), \( N(t) = N(t_0) \), then
\[
C = \frac{\log((b+\delta) - dN(t_0))}{-d} - t_0 \cdot \cdot.
\]

Putting the value of \( C \) in equation (4.3) we get,
\[
N(t) = \frac{1}{d} \left[ (b+\delta) - (b+\delta) dN(t_0) e^{-d(t-t_0)} \right].
\]

Thus,
\[
\lim_{t \to \infty} N(t) = \frac{b+\delta}{d}.
\]

This is the conclusion of this theorem.
It is clear that the limit set of the system (2.2) is on the plane \( S + I + R = \frac{b + \delta}{d} \). Thus we focus on the reduced system

\[
\frac{dl}{dt} = \frac{dkl}{d + \alpha(b + \delta) + (\beta - \alpha)dl - \alpha dR} \left( \frac{b + \delta}{d} - I - R \right) - (d + \mu + \theta)l = P(I, R),
\]

(4.4)

And

\[
\frac{dR}{dt} = \mu I - (d + \gamma)R \equiv Q(I, R),
\]

(4.5)

We have the following result regarding the non existence of periodic orbits in system (4.4) which implies the non-existence of periodic orbits of system (2.2) by theorem 1.

**Theorem 2.** Then System (4.4)–(4.5) does not have non trivial periodic orbits, if\((2d + \gamma + \mu + \theta)(\beta - \alpha) > \mu \alpha\).

**Proof:** Consider system (4.4)–(4.5) for, \( I > 0, R > 0 \). Take a Dulac function

\[
D(I, R) = \frac{1}{(d + \alpha(b + \delta) + (\beta - \alpha)dl - \alpha dR)}
\]

We have

\[
\frac{\partial(DP)}{\partial I} + \frac{\partial(DQ)}{\partial R} = -1 - \frac{(d + \gamma)(d + \alpha + (b + \delta))}{dkl}
\]

\[
- \frac{1}{k} \left[ (2d + \gamma + \mu + \theta)(\beta - \alpha) - \mu \alpha < 0, \right.
\]

if\((2d + \gamma + \mu + \theta)(\beta - \alpha) > \mu \alpha\),

To study the proportion of the Disease free equilibrium \( E_0 \) and endemic equilibrium \( E_1 \), we rescale the system (4.4)–(4.5) as,

We consider the reduced system as

\[
x = \frac{k}{d + \gamma} I, \quad y = \frac{k}{d + \gamma} R, \quad \tau = (d + \gamma) t.
\]

Then

\[
\frac{dx}{d\tau} = \frac{d}{d + \gamma} \left( \frac{kl}{d + \gamma} \right) = \frac{k}{d + \gamma} \left( \frac{dl}{dt} \frac{dt}{d\tau} \right)
\]

\[
\frac{dx}{d\tau} = \frac{p(x - y) - mx}{1 + qx - ry}
\]

(4.6)

where

\[
p = \frac{d}{d + \alpha(b + \delta)}, \quad A = \frac{k(B + \delta)}{d(d + \gamma)}, \quad q = \frac{(d + \gamma)(\beta - \alpha)d}{d + \alpha(b + \delta)}
\]

\[
r = \left( \frac{(d + \gamma)\alpha\delta}{d + \alpha(b + \delta)} \right)^k, \quad M = \frac{d + \mu + \theta}{d + \gamma},
\]

and

\[
\frac{dy}{d\tau} = \frac{dy}{dt} \frac{dt}{d\tau}
\]

\[
= \frac{1}{(d + \gamma)} \frac{d}{dt} \left( \frac{kR}{d + \gamma} \right),
\]

\[
\frac{dy}{d\tau} = \frac{k}{d + \gamma} \frac{kl}{d + \gamma} \frac{kR}{d + \gamma} = sx - y.
\]

(4.7)

where \( s = \frac{\mu}{d + \gamma} \),

\[
\]
The disease free equilibrium \( E_0(0,0) \), of system (4.6) - (4.7) is the DFE of system (2.2) and the endemic equilibrium \( E_1\left( x^*, y^* \right) \) of the system (4.6) - (4.7) is the endemic equilibrium of model (2.2). For this equation (4.6),

\[
\begin{align*}
\rho (Ax - y) - m &= 0 \\
\rho y - m &= \rho(1 + s)(q - rs) x,
\end{align*}
\]

\[
x^* = \frac{\rho A - m}{\rho(1 + s) + m(q - rs)}.
\]

By equation (4.6),

\[
y^* = sx^*.
\]

The Jacobian matrix of system (4.6) - (4.7) at DFE \( E_0 \) is

\[
J_{E_0} = \begin{bmatrix} A\rho - m & 0 \\ s & -1 \end{bmatrix}.
\]

Its characteristics equation

\[
(A\rho - m - \lambda)(-1 - \lambda) = 0
\]

and its characteristic roots are \( \lambda = -1, \lambda = -(m - A\rho) \).

If \( m - A\rho > 0 \) then the DFE \((0,0)\) of, system (4.6) - (4.7) is a stable hyperbolic node.

If \( m - A\rho = 0 \) then the DFE is a saddle node.

If \( m - A\rho < 0 \) then the DFE is a hyperbolic saddle node.

The Jacobian matrix of system (4.6) - (4.7) at endemic equilibrium \( E_1\left( x^*, y^* \right) \) is

\[
J_{E_1} = \begin{bmatrix} \frac{\rho y^* r + q - (1 + Aq)}{1 + q x^* - ry^*} & \frac{\rho (Ar - 1) - x^* (q + r)}{1 + q x^* - ry^*} \\ s & -1 \end{bmatrix}.
\]

Let \( \text{tr}(J_{E_1}) = \frac{\rho y^* r + q - (1 + Aq)}{1 + q x^* - ry^*} - 1 \),

\[
= \frac{-\rho(-1 - s - A(q - rs))}{1 + q x^* - ry^*}.
\]

Since \( q > rs \) then, \( J_{E_1} > 0 \).

Also \( \text{tr}(J_{E_1}) = \frac{\rho y^* r + q - (1 + Aq)}{1 + q x^* - ry^*} - 1 \),

\[
= \frac{\rho y^* (r + q) - \rho(A + Aq) - x^* (q + r)}{1 + q x^* - ry^*}.
\]

The sign of \( \text{tr}(J_{E_1}) \) is obtained by

\[
S_1 = \rho y^* (r + q) - \rho(A + Aq).
\]

Putting \( x^* = \frac{\rho A - m}{\rho(1 + s) + m(q - rs)} \), we have

\[
S_1 = \frac{\rho(r + q)(\rho A - m) - \rho(A + Aq)[\rho(1 + s) + m(q - rs)]}{\rho(1 + s) + m(q - rs)}.
\]
Hence
\[ S_1 < 0 \] if \( \mu A - m > 0 \) or \( m - A \rho < 0 \),
so
\[ \nu(E_1) < 0 \] if \( m - A \rho < 0 \).
The endemic equilibrium \( E_1(x^*, y^*) \) is asymptotically stable and it is globally stable if \( R_0 > 1 \).

V Example for Justification
We take the parameter of the system as \( d = 2.29, \alpha = 3.1, \beta = 4.1, b = 3.1, \theta = 0.3, \gamma = 0.49, A = 6.4, \mu = 0.19, \delta = 0.2 \), and \((S(0),I(0),R(0)) = (4.1.1)\). Then \( E_0 = (1.441, 0, 0) \) and \( R_0 = 0.72051 < 1 \).
Therefore the DFE \( E_0 \) is globally stable in first octant. Hence the disease dies out.

Again, if \( d = 0.29, \alpha = 1.1, \beta = 4.7, b = 3.1, \gamma = 1.5, k = 6.5, \theta = 0.3, \mu = 0.19, \delta = 0.2 \), \((S(0),I(0),R(0)) = (4.1.1)\)
Then \( E^*(x^*, y^*, R^*) = (3.56343, 68330.3909) \) and \( R_0 = 59.733 > 1 \). Therefore the endemic equilibrium \( E_1 \) is globally stable. Hence the disease is endemic.

VI Conclusion
Mathematical modeling is useful tool for understanding dynamic of infectious diseases. The paper proposes SIRS epidemic model with immigration type incidence. We obtain a disease free equilibrium which is globally stable if \( R_0 < 1 \) and an endemic equilibrium which is locally stable if \( R_0 > 1 \). Also we give an example for validity of our results.

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REFERENCES