Fixed Point for Expansion Mappings in Cone b-Metric Space

R.K. VERMA, Department of Mathematics, Govt. C.L.C. College Patan, Distt.-Durg (C.G.), PIN-491111, INDIA, e-mail: rohitverma1967@rediffmail.com

Abstract. In this paper, we introduce T-extension condition for expansive mappings, and prove some common fixed point results for expansive type mappings in the cone b-metric spaces. In the first part of the paper, we will prove a general common fixed point theorem for compatible mappings. Then we introduce the T-extension condition in cone b-metric spaces. Suppose S:X→X be a self-mapping in a cone b-metric space and it has a fixed point but it does not satisfy an expansive condition. Note that T-extensions mappings are those mappings, which make the given self-mapping S:X→X to be an expansive type mappings so that the common fixed point can be extracted from S and T. More precisely, there are some mappings, say S:X→X, which have a fixed point but do not satisfy an expansive condition. If another mapping T: X→X is introduced, then expansive condition holds. In this case S is said to satisfy T-extension condition. In the second part, we will prove a common fixed point theorem for self-mapping S:X→X, with mapping T:X→X under T-extensions satisfying an expansive type conditions in the cone b-metric space. We have validated our theorems by examples.

Key words and phrases: Cone metric space; b-metric space; Cone b-metric space; weakly compatible mappings; contraction mappings; expansion mappings.

I. INTRODUCTION

In 1922, Banach [10] proved a famous fixed point theorem for metric space, known as Banach contraction principal. It establishes the existence and uniqueness of a solution of an operator equation Tx = x. Since then, there are many generalizations of Banach’s contraction mapping principle in the fixed point literature. These generalizations are two folded, e.g., either by using a (relaxed) contractive condition on set X, or by imposing extra conditions on space X. Since then many fixed and common fixed point theorems are proved in the various spaces (e.g., [12], [36], [33], [43], [47], [52], [58], [60], [61] etc.)

In 2007, Huang and Zhang [23] introduced the cone metric space. In this space, each pair of points is assigned to a member of real Banach space E with a cone. A similar notion was also considered by Rzepecki in [46]. After carefully defining convergence and completeness in cone metric spaces, Alghamdi et al. [7] proved some fixed point theorems of contractive mappings. Another fixed point results in cone metric spaces appeared in [39], [56]. In 2010, the topological questions in cone metric spaces were studied by Turkoglu and Abuloha [56], where it was proved that every cone metric space is a first-countable topological space. Hence, continuity is equivalent to sequential continuity and compactness is equivalent to sequential compactness. In their work, with the structure of a cone b-metric space, they [56] established some topological properties of the cone b-metric spaces. Actually, they extended some results of Khamisi and Hussain [31].

Similarly, the b-metric space, introduced by Bakhtin [11] is another generalization of the metric space. In 1993, Czerwik [16] expanded Banach’s fixed point theorem in b-metric space. Cone metric space was introduced by Huang and Zhang [23] in 2007. While studying the cone metric spaces, Khamisi [30] re-introduced the b-metric space to which he gave the name of metric-type space. Several papers have been published in metric-type spaces which contains fixed point results for single valued and multi-valued functions (e.g., [3], [6], [27]). Alghamdi et al. [New-36] introduced the notion of b-metric like space which generalized the notion of b-metric space, where they proved some exciting new fixed point results in b-metric like space. M.A. Alghamdi et al. [7], gave the concept of dislocated quasi b-metric space, which was a generalization of dislocated quasi metric, partial b-metric, b-dislocated metric and b-metric spaces. Fixed point results for Banach’s contraction principle, Kannan and Chetterjae type fixed point results for self-mapping in such a space is established by them [7].

The cone b-metric space is a generalization of both cone metric space and b-metric space. This space was given by Hussain and Shah [25] in 2011. In this paper, they established some topological properties of the cone b-metric spaces and then improved some recent results about KKM mappings in the setting of a cone b-metric space. They also proved some fixed point existence results for multi-valued mappings defined on these spaces.

Some papers in cone b-metric spaces are: Saleh et al. [47], Rahimi, Vetro and Rad [42], Dubey, Verma and Dubey [18], Azam and Arsad [9], Anil Kumar and Rathee [2], Rezapour and Hamlbarani [44] and Janković et al.[26] etc. Some papers on b-metric spaces are: Mohanta [35], Demmaa et al. [20], etc. Some papers on cone b-metric spaces are: Rima Maitra [34], Huang and Xu [21], Huang, Zhu and Xi-Wen [24] proved a fixed point theorem for cone b-metric space using expansion type mappings. S.K. Mohanta [35] proved coincidence points and common fixed points for expansive type mappings in b-metric spaces. The concept of compatible mapping was given by Jungck [28] in 1986. This was further generalized to weakly compatibility by Jungck [29], in 1996.
Ordered normed spaces, normal cones and topical functions ([11], [23], [32], [36], [44]) have some applications in optimization theory. An order is introduced by using vector space cones. Huang and Zang [23] used this approach and they have replaced the real numbers \( \mathbb{R} \) by an ordered Banach space \( E \) and defined a cone metric space. For fixed point results in cone metric and b-metric space see [3], [4], [5], [8], [32], [33], [41], [44], [45], [48], [49], [50], [51], [53], [54], [57].

In 2013, Popovic, Radenovic and Shukla [40], introduced the concept of tvs-cone b-metric space over solid cone. Their results improve, generalize, extend, unify, enrich and complement some recent fixed point results established by H. Huang and S. Xu [22], M.P. Stanic et al. [39], A.S. Cvetkovic et al. [17], M.H. Shah et al. [49] and references therein.

In this paper, we prove some common fixed point results for expansive type mappings in the cone b-metric spaces. In the first part of the paper, we will prove a general common fixed point theorem for compatible mappings satisfying an expansive condition in cone b-metric spaces. Further, we introduce the T-extension conditions in cone b-metric spaces. Suppose \( S:X \rightarrow X \) be a self mapping in a cone b-metric space and have a fixed point in it but it does not satisfy an expansive condition. Note that T-extensions mappings are those mappings, which make the given self-mapping \( S:X \rightarrow X \) to be an expansive type mappings so that the common fixed point can be extracted from \( S \) and \( T \). More precisely, there are some mappings, say \( S:X \rightarrow X \), which have a fixed point but do not satisfy an expansive condition. If another mapping \( T:X \rightarrow X \) is introduced, then expansive condition holds. In this case \( S \) is said to satisfy \( T \)-extension condition. In the second part, we will prove a common fixed point theorem for self-mapping \( S:X \rightarrow X \), with mapping \( T:X \rightarrow X \) under T-extensions satisfying an expansive type conditions in the cone b-metric space. We have validated our theorems by examples.

Now, we give definitions regarding above notations.

**Definition 1.1** Let \( X \) be a non-empty set and \( \leq \) be a relation defined in it. Then \( (X, \leq) \) is called a partial order set (POSET) if

- (POSET-1) \( \leq \) is reflexive, i.e., \( x \leq x, \forall x \in X \)
- (POSET-2) \( \leq \) is anti-symmetric, i.e., \( x \leq y \) and \( y \leq x \) \( \Rightarrow x = y \), \( \forall x, y \in X \)
- (POSET-3) \( \leq \) is transitive, i.e., \( x \leq y \) and \( y \leq z \) \( \Rightarrow x \leq z \), \( \forall x, y, z \in X \)

**Definition 1.2** ([23]) Let \( E \) be a complete normed linear space and \( P \) be a nonempty subset of \( E \). Then \( P \) is called cone if and only if

- (C-1) \( P \) is closed and \( P \neq \emptyset \), where \( \emptyset \) is zero element of \( E \).
- (C-2) \( \forall x, y \in P \) and \( a, b \geq 0 \), \( ax+by \in P \)
- (C-3) \( P \cap (-P) = \emptyset \).

Using above definition of cone \( P \), one can define a partial order relation \( \leq \) in \( P \) satisfying: (PO-1) \( x \leq y \Leftrightarrow y-x \in P \);

- (PO-2) \( x \ll y \Leftrightarrow x \leq y \) and \( x \neq y \);
- (PO-3) \( x \ll y \) means \( y-x \in P \), \( \forall x, y \in P \).

Cone \( P \) is called normal, if \( \exists K > 0 \) such that:

- (PO-4) \( \theta \leq x \ll y \Rightarrow ||x|| \leq K||y|| \), \( \forall x, y \in P \).

The least positive real number satisfying condition (iv) is called normal constant of \( P \). Note that, for normal cone \( K \geq 1 \).

The cone \( P \) is called regular if every increasing sequence which is bounded above is convergent. Equivalently, the cone \( P \) is called regular if every decreasing sequence which is bounded below is convergent. Every regular cone is normal but there exist normal cones which are not regular. In 2007, using properties of cones and partial ordering conditions (PO-1)-(PO-2), Huang and Zhang [23] defined the cone metric \( d:X \times X \rightarrow E \), in the following way:

**Definition 1.3** ([23]) Let \( X \) be a non-empty set. Define \( d:X \times X \rightarrow E \) satisfying

- (CM-1) \( \theta \leq d(x, y), \forall x, y \in X \), and \( d(x, y) = \theta \) if and only if \( x = y \), \( \forall x, y \in X \);
- (CM-2) \( d(x, y) = d(y, x), \forall x, y \in X \);
- (CM-3) \( d(x, y) \leq d(x, z)+d(z, y) \), \( \forall x, y, z \in X \).

Then \( (X, d) \) is called cone metric space.

In 2011, using the definition of b-metric space of Cz’erviic’ [16], Hussain and Shah [25] introduced the notion of cone b-metric space in the following way:

**Definition 1.4** ([25]) Suppose \( X \) be a non-empty set and \( b \geq 1 \) is constant. Define mapping \( d:X \times X \rightarrow E \). Then \( (X, d) \) is said to be cone b-metric space if:

- (CbM-1) \( \theta \leq d(x, y), \forall x, y \in X \), and \( d(x, y) = \theta \) if and only if \( x = y \), \( \forall x, y \in X \);
- (CbM-2) \( d(x, y) = d(y, x), \forall x, y \in X \);
- (CbMS-3) \( d(x, y) \leq b[d(x, z)+d(z, y)], \forall x, y, z \in X \).
Note that every cone metric space is a cone b-metric space (with $b=1$), but the converse need not be true, as shown in following example.

**Example 1.5** ([21]) Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$, $P = \{(x, y) \in E: x \geq 0, y \geq 0\}$. Define $d: X \times X \to E$ by:

$$d(x, y) = (|x-y|, |x-y|)$$

Then $(X, d)$ is a cone b-metric space with coefficient $b = 6/5 > 1$. But it is not a cone metric space since the triangle inequality does not satisfy. We observe that,

1. $(1, 1) = d(1, 2) > d(1, 4) + d(4, 2) = (1/3, 1/3) + (1/2, 1/2) = (5/6, 5/6)$,
2. $(1, 1) = d(3, 4) > d(3, 1) + d(1, 4) = (1/2, 1/2) + (1/3, 1/3) = (5/6, 5/6)$.

Thus $(X, d)$ is not a cone metric space.

**Example 1.6** ([49]) Let $X$ be the set of Lebesgue measurable functions on $[0, 1]$ such that

$$\int_0^1 |f(x)|^2 \, dx < \infty$$

Define $d: X \times X \to [0, \infty]$ by

$$d(f, g) = \frac{1}{2} \int_0^1 |f(x) - g(x)|^2 \, dx$$

Then, we observe that $d$ satisfies the following properties:

(i) $d(f, g) = 0 \iff f = g$

(ii) $d(f, g) = d(g, f)$

(iii) $d(f, g) \leq 2[d(f, h) + d(h, g)]$, $\forall f, g, h \in X$.

So that, $(X, d)$ is a cone b-metric space with constant $b=2$.

Mohamed A. Khamsi [30] has shown this example for metric type space. We refer to Ex.1.4 and Ex.1.5 of Huang and Xu [21] for cone b-metric space, which are not b-metric space. Following is an example of cone metric space.

**Example 1.7** ([18], [19]) Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E: x,y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}$ and $d: X \times X \to E$ such that $d(x, y) = (|x-y|, \infty|x-y|)$, where $\infty \geq 0$ is a constant. Then $(X, d)$ is a cone metric space.

We refer to Ex.1.3 and Ex.1.4 of Tiwari et.al.[55] and Ex.2 of Huang and Xian [23], for cone metric space.

**Example 1.8** ([42]) Let $B = \{e_i : i=1, 2, 3, \ldots, n\}$ be an orthonormal basis of $\mathbb{R}^n$ with inner product $(\cdot, \cdot)$ and $p>0$. Define the space

$$X_p = \{[x]: x \in [0, 1] \to \mathbb{R}^n, \int_0^1 \|x(t), e_j\|^p dt \in \mathbb{R}, \text{where } j = 1, 2, 3, \ldots, n\}$$

where $[x]$ represents the class of equivalence of $x$ with respect to relation of functions equal almost everywhere. Let $E = \mathbb{R}^n$ and $P = \{y \in \mathbb{R}^n: (y, e_i) \geq 0, i=1, 2, 3, \ldots\}$ be a solid cone.

Define $d(f, g) = \sum_{i=1}^{n} |(f(e(t), e_i), e_j)|^p dt$, $f, g \in X_p$.

Then $(X_p, d)$ is a cone metric space with $K = 2^{p-1}$. If $p>1$ then $(X_p, d)$ is cone b-metric space.

**Definition 1.9** ([25]) Let $(X, d)$ be a cone b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then

(i) $\{x_n\}$ converges to $x$ whenever, for every $c \in E$ with $0 \ll c$, there is a natural number $N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$).

(ii) $\{x_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $0 \ll c$, there is a natural number $N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) $(X, d)$ is a complete cone b-metric space if every Cauchy sequence is convergent.

The following lemmas are often used (in particular when dealing with cone metric space in which the cone need not be normal).

**Lemma 1.10** ([26]) Let $P$ be a cone and $\{a_n\}$ be a sequence in $E$. If $c \in \text{int}P$ and $\theta \leq a_n \to \theta$ (as $n \to \infty$), then there exists $N$ such that for all $n > N$, we have $a_n \ll c$.

**Lemma 1.11** ([26]) Let $x, y, z \in E$, if $x \ll y$ and $y \ll z$, then $x \ll z$. 

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Lemma 1.12 (\cite{25}) Let $P$ be a cone and $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$, then $u = \theta$.

Lemma 1.13 (\cite{15}) Let $P$ be a cone. If $u \in P$ and $u \ll ku$ for some $0 \leq k < 1$, then $u = \theta$.

Lemma 1.14 (\cite{26}) Let $P$ be a cone and $a \leq b + c$ for each $c \in \operatorname{int} P$, then $a \leq b$.

Definition 1.15 (\cite{15}) Let $S, T : X \to X$ be two self-mapping of a metric space $(X, d)$ and $\{x_n\}$ be a sequence in $X$. Mappings are called compatible if there exist $t \in X$ such that $\lim_{n \to \infty}d(STx_n, TSx_n) = 0$, whenever $\lim_{n \to \infty}Sx_n = \lim_{n \to \infty}Tx_n = t$.

Definition 1.16 (\cite{29}) Let $S, T : X \to X$ be two self mapping of a metric space $(X, d)$. The mappings are called weakly compatible if they commute at their coincidence point. That is, $STu = TSu$, whenever $Su = Tu$, where $u \in X$.

Definition 1.17 Let $(X, d)$ be a metric space and $T : X \times X \to E$. Then $T$ is said to be Banach $k$-contraction if, for all $x, y \in X$, there exist $k < 1$, satisfying $d(Tx, Ty) < kd(x, y)$.

Ozturk and Kaplan \cite{38} shown that, there are mappings which are not contraction itself, but it holds “contraction condition” by introducing some mapping $T : X \to X$. This condition was called $T$-contraction.

Definition 1.18 (\cite{13}, \cite{14}, \cite{37}) Let $T$ and $S$ be two self mappings on a metric space $(X, d)$. A mapping $S$ is said to be $T$-contraction if there exist a real constant $k \in (0, 1)$ such that:

\[ d(TSx, TSy) \leq kd(Tx, Ty), \]

for all $x, y \in X$.

An example can be obtained regarding this in the paper of Ozturk and Kaplan \cite{38}. This shows the importance of $T$-contraction condition in the arena of fixed point theorem. Following example also shows that a mapping $S : X \to X$ does not satisfy $k$-contraction condition itself but if $T : X \to X$ is introduced, then it satisfy $T$-contraction. Here, note that $S$ has a fixed point:

Example 1.19 Let $X = [1, +\infty)$ with the metric induced in $\mathbb{R}$. Define $d : X \times X \to \mathbb{R}$ by:

\[ d(x, y) = |x - y|. \]

Let $S : X \to X$ be a self-mapping defined by:

\[ Sx = \frac{4}{\sqrt{x}} , \]

for all $x \in X$. We will claim that $S$ does not satisfy the Banach $k$-contraction condition $d(Sx, Sy) \leq kd(x, y)$, $k < 1$. Since

\[ d(Sx, Sy) = \left| \frac{4}{\sqrt{x}} - \frac{4}{\sqrt{y}} \right| = \frac{4d(x, y)}{\sqrt{xy}(\sqrt{x} + \sqrt{y})} \]

is contraction $\iff \sqrt{xy}(\sqrt{x} + \sqrt{y}) \leq 8.$

Now, introduce mapping $T : X \times X \to \mathbb{R}$ defined by $Tx = 1 + \ln x$, for all $x$ in $X$. Then $S$ is a $T$-contraction, satisfying:

\[ d(TSx, TSy) = |1 + \ln(Sx) - 1 - \ln(Sy)| = |\ln \left( \frac{4}{\sqrt{x}} \right) - \ln \left( \frac{4}{\sqrt{y}} \right)| = |\frac{1}{2} \ln x - \frac{1}{2} \ln y| = \frac{1}{2} d(Tx, Ty). \]

Showing that, $S$ is a $T$-contraction, though it is not a contraction. Also $x = 4$ is unique common fixed point in $X$.

Inspiring by Definition 1.17, we define the expansion mapping below:

Definition 1.20 Let $(X, d)$ be a metric space and $T : X \times X \to E$. Then $T$ is said to be an expansion mapping if, for all $x, y \in X$, there exist $\eta > 1$, satisfying $d(Tx, Ty) \geq \eta d(x, y)$.

Keeping $T$-contraction in consideration, we now introduce $T$-extension condition, as follows:

Definition 1.21 Let $S : X \to X$ be a self-mapping of a cone $b$-metric space $(X, d)$, with constant $b \geq 1$. The mapping $S$ is said to satisfy $T$-extension, with respect to the injective and continuous mapping $T : X \to X$, if the following inequality satisfies:

\[ d(Tx, Ty) \geq \eta d(TSx, TSy), \]

for all $x, y \in X$, where $\eta > b$.

Example 1.22 Let $X = [1, +\infty]$ with the metric induced in $\mathbb{R}$. Define metric $d : X \times X \to \mathbb{R}$ by:

\[ d(x, y) = |x - y|. \]

Let $S : X \to X$ be a self-mapping defined by:

\[ Sx = \frac{4}{\sqrt{x}} , \]

for all $x \in X$. We will claim that $S$ does not satisfy $\text{extension}$ condition $d(Sx, Sy) \geq \eta d(x, y)$, where $\eta > 1$. Since

\[ d(Sx, Sy) = \left| \frac{4}{\sqrt{x}} - \frac{4}{\sqrt{y}} \right| = \frac{4d(x, y)}{\sqrt{xy}(\sqrt{x} + \sqrt{y})} \leq \frac{4d(x, y)}{\sqrt{(xy)^{3/4}}} \]

(using A.M. $\leq$ G.M.)

$\iff$

\[ d(Sx, Sy) < 2d(x, y), \]

for all $x, y \in X$, with $x \neq y$.

Thus $S$ does not satisfy extension condition for $\eta = 2$. On the other hand, by introducing $T : X \times X \to \mathbb{R}$ defined by $Tx = 1 + 4ln x$; we observe that $S$ is $T$-contraction. Here

\[ d(TSx, TSy) = |1 + 4\ln(Sx) - 1 - 4\ln(Sy)| = 4|\ln(4/\sqrt{x}) - \ln(4/\sqrt{y})| = 4|\frac{1}{2}\ln x - \frac{1}{2}\ln y| = \frac{1}{2} |4\ln x - 4\ln y| = \frac{1}{2} d(Tx, Ty). \]

Thus, $d(Tx, Ty) = 2d(TSx, TSy)$. Hence $S$ is $T$-extension for $\eta = 2$.

It shows that $S$ is a $T$-extension, though it is not an extension. Also $x = 2$ is unique common fixed point in $X$.  

II. PRELIMINARIES

In 1984, Wang et al. [59] presented fixed point result for expansive mappings in metric spaces which correspond to some contractive mapping in [45]. Zhang [62] has done remarkable work in this field. In order to generalize the results of fixed point theory, he published his work, in which the fixed point problem for expansive mapping is systematically presented in a chapter. As applications, he also investigated the existence of solutions of equations for locally condensing mapping and locally accretive mapping.

Using the extension condition, Xi-Wen [24] proved the following Theorem (Cor. 2.1, p.144):

**Theorem 2.1** ([24]) Let \((X, d)\) be a complete cone metric space and \(T:X \to X\) be a surjection. Suppose that there exist \(k > 1\) satisfying:

\[ d(Tx, Ty) \geq kd(x, y), \text{ for all } x, y \in X. \]

Then \(T\) has a unique fixed point.

Above theorem was the generalization of the main result (Theorem 2.1) of Xi-Wen:

**Theorem 2.2** ([24]) Let \((X, d)\) be a complete cone metric space and \(T:X \to X\) be a surjection. Suppose that there exist \(a_1\), \(a_2\), \(a_3 \geq 0\), with \(a_1 + a_2 + a_3 > 1\), satisfying expansion condition:

\[ d(Tx, Ty) \geq a_1d(x, y) + a_2d(Tx, Ty) + a_3d(y, Ty), \text{ for all } x, y \in X, x \neq y. \]

Then \(T\) has a unique fixed point.

In this paper, we generalize above results for two self-mappings. We prove following Lemma, which is useful for our main results:

**Lemma 2.3** Let \((X, d)\) be a cone \(b\)-metric space and \(\{x_n\}\) be a sequence in \(X\). If there exists a number \(k \in (0, 1/b)\), where \(b \geq 1\) such that

\[ d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}), n = 1, 2, \ldots \]

then \(\{x_n\}\) is a Cauchy sequence in \(X\).

**Proof.** By the simple induction with the condition (1), we have

\[ d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}) \leq k^2d(x_{n-1}, x_{n-2}) \leq \ldots \leq k^nd(x_1, x_0). \]

For any integers \(m \geq 1, p \geq 1,\) it follows that

\[
\begin{align*}
d(x_{m+p}, x_m) & \leq b[d(x_{m+p}, x_{m+p-1}) + d(x_{m+p-1}, x_m)] = bd(x_{m+p}, x_{m+p-1}) + bd(x_{m+p-1}, x_m) \\
& \leq bd(x_{m+p}, x_{m+p-1}) + b^2[d(x_{m+p-1}, x_{m+p-2}) + d(x_{m+p-2}, x_m)] \\
& = bd(x_{m+p}, x_{m+p-1}) + b^2d(x_{m+p-1}, x_{m+p-2}) + b^2d(x_{m+p-2}, x_m) \\
& \leq bd(x_{m+p}, x_{m+p-1}) + b^2d(x_{m+p-1}, x_{m+p-2}) + b^2d(x_{m+p-2}, x_{m+p-3}) + \ldots + b^{p-1}d(x_{m+2}, x_{m+1}) + b^{p-1}d(x_{m+1}, x_m) \\
& \leq bk^{m+p-1}d(x_1, x_0) + b^2k^{m+p-2}d(x_1, x_0) + b^3k^{m+p-3}d(x_1, x_0) + \ldots + b^pk^{m+1}d(x_1, x_0) + b^n\theta d(x_1, x_0) \\
& = [bk^{m+p} + b^2k^{m+p-1} + b^2k^{m+p-2} + \ldots + b^pk^{m+1}]d(x_1, x_0) + \theta d(x_1, x_0) \\
& \leq b^k \theta d(x_1, x_0)/(b-k) + b^{k-1}\theta d(x_1, x_0)
\end{align*}
\]

Let \(\theta \ll c\) be given. Note that, the right hand side term \(b^k \theta d(x_1, x_0)/(b-k) + b^{k-1}\theta d(x_1, x_0)\to \theta \) as \(m \to \infty\) for any \(k\).

Using Lemma 1.10, we find that there exist \(m_0 \in \mathbb{N}\) such that

\[ b^k \theta d(x_1, x_0)/(b-k) + b^{k-1}\theta d(x_1, x_0) \ll c, \text{ for each } m > m_0. \]

Thus,

\[ d(x_{m+p}, x_m) \leq [b^k \theta (b-k)]d(x_1, x_0) + b^{k-1}\theta d(x_1, x_0) \ll c, \text{ for all } m > m_0 \text{ and any } p \in \mathbb{N}. \]

So, by Lemma 1.11, \(\{x_n\}\) is a Cauchy sequence in \((X, d)\). This completes the proof.

III. MAIN RESULTS

A. Fixed point for expansion mapping

In this subsection, we give our first main result below:

**Theorem 3.1** Let \((X, d)\) is a cone \(b\)-metric space with constant \(b \geq 1\). Suppose \(S, T: X \to X\) are two self-mappings, satisfying the expansion inequality:

\[ d(Tx, Ty) \geq \lambda d(Sx, Sy) + \mu[d(Sx, Tx)+d(Sy, Ty)], \forall x, y \in X, x \neq y, \]

\[ (3.1) \]
where \([(1-\mu)/(\lambda+\mu)] \in (0, 1/b); \lambda+\mu > 0, \lambda+2\mu > 1 \text{ and } 0 < \mu < 1. If S(X) \subseteq T(X) and one of S(X) or T(X) is complete, then S and T have a point of coincidence. Moreover, if the pair (S, T) is weakly compatible, then S and T have a unique common fixed point in X.

Proof. Since \( b \geq 1 \) and \([(1-\mu)/(\lambda+\mu)] \in (0, 1/b) \) with \( \lambda+\mu > 0, \lambda+2\mu > 1 \) and \( 0 < \mu < 1 \). If \( S(X) \subseteq T(X) \) and one of \( S(X) \) or \( T(X) \) is complete, then \( S \) and \( T \) have a point of coincidence. Moreover, if the pair \( (S, T) \) is weakly compatible, then \( S \) and \( T \) have a unique common fixed point in \( X \).

Choose \( x_0 \in X \). Since \( S(X) \subseteq T(X) \), there exist \( x_1 \in X \) such that \( Sx_0 = Tx_1 = y_1 \) (say). For this \( x_1 \); \( Sx_1 = Tx_2 = y_2 \) (say).

Continuing in this manner, we have
\[
Sx_n = Tx_{n+1} = y_{n+1} \quad (3.2)
\]
Choose \( y_n \neq y_{n+1} \), so that \( d(y_n, y_{n+1}) > 0 \). Now, putting \( x=x_n, y=x_{n+1} \) in (3.1), we have
\[
d(Tx_n, Tx_{n+1}) \geq \lambda d(Sx_n, Sx_{n+1}) + \mu[d(Sx_n, Tx_n)+d(Sx_{n+1}, Tx_{n+1})]
\]
\[\text{i.e., } d(y_n, y_{n+1}) \geq \lambda d(y_{n+1}, y_{n+2}) + \mu[d(y_{n+1}, y_n)+d(y_{n+2}, y_{n+1})]
\]
or,
\[
(1-\mu) d(y_n, y_{n+1}) \geq (\lambda+\mu) d(y_{n+1}, y_{n+2}) \quad (3.3)
\]
Similarly, by putting \( x = x_{n+1} \) and \( y = x_n \) in (3.2), we get the same inequality as in (3.3).

Thus
\[
d(y_{n+2}, y_{n+1}) \leq \left[\frac{1-\mu}{\lambda+\mu}\right] d(y_n, y_{n+1}), \text{ for all } n \in \mathbb{N} \quad (3.4)
\]
Since \( \mu < 1 \) and \( \lambda+\mu > 0 \), we have \([(1-\mu)/(\lambda+\mu)] \in (0, 1/b) \), where \( b \geq 1 \). This is true, because
\[(1-\mu)/(\lambda+\mu) > 0 \text{ and } \lambda+\mu > 0, \text{ gives } (1-\mu) > 0; \text{ and therefore}
\[(1-\mu)/(\lambda+\mu) < 1/b \]
whence \( \lambda+\mu > 0 \), which is true. Thus \([(1-\mu)/(\lambda+\mu)] < 1/b \) is valid.

So, let us take \( k = [(1-\mu)/(\lambda+\mu)] \in (0, 1/b) \). Hence, from Lemma 2.3, the sequence \( \{y_n\} \) satisfying the inequality (3.4) is a Cauchy sequence in \( X \). Further, if \( y_n \in S(X) \), then since \( S(X) \subseteq T(X) \) and \( T(X) \) is complete, so \( \{y_n\} \) converges to some \( z \in T(X) \). Similarly, if \( y_n \in T(X) \) and \( T(X) \) is complete, then \( \{y_n\} \) converges to some \( z \in T(X) \). Thus, in both cases, \( \{y_n\} \) converges to \( z \in T(X) \). Let \( z = Tu \), for some \( u \in X \). So that
\[
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = \lim_{n \to \infty} y_n = z = Tu \in T(X) \quad (3.5)
\]
We claim that \( Su = z \). If not, then \( d(z, Su) > 0 \). Now, putting \( x = x_n, y = u \) in (3.1), we have
\[
d(Tx_n, Tu) \geq \lambda d(Sx_n, Su) + \mu[d(Sx_n, Tu)+d(Su, Tx_n)];
\]
which implies, as \( n \to \infty \) and using (3.5), that
\[\theta \geq \lambda d(z, Su) + \mu[d(z, z)+d(Su, z)], \text{ i.e., } \theta \geq (\lambda+\mu) d(z, Su); \text{ a contradiction. So that } Su = z.
\]
Thus \( u \) is a coincidence point of \( (S, T) \). Similar argument arises if we assume that \( S(X) \) is complete. In this case \( \{y_n\} \) will converge to some \( z \in S(X) \), and we may use \( z=Su \) instead of \( z=Tu \) as in (3.5). The inequality (3.1) will give \( z = Su \). Thus in both cases, \( u \in X \) is a coincidence point of \( (S, T) \). This proves first part of the theorem.

Further, if the pair \( (S, T) \) is weakly compatible, then \( STu = Tu \) or \( Sz = Tz \). We claim that \( z \) is a common fixed point of \( S \) and \( T \). If not, then \( d(z, Sz) > 0 \). Now, putting \( y = z, x = x_n \) in (3.1), we have
\[
d(Tx_n, Tz) \geq \lambda d(Sx_n, Sz) + \mu[d(Sx_n, Tz)+d(Sz, Tx_n)];
\]
which implies, as \( n \to \infty \) and using (3.5),
\[d(z, Sz) \leq \lambda d(z, Sz) + \mu[d(z, Sz)+d(z, z)], \text{ i.e., } \]
\[d(z, Sz) \leq (\lambda+\mu) d(z, Su), \text{ or, } [(1-\mu)/(\lambda+\mu)] \geq 1, \text{ (as } d(z, Su) > 0)\]
a contradiction of \( k = [(1-\mu)/(\lambda+\mu)] < 1 \). Thus \( Sz = z \). Hence \( z \) is a common fixed point of \( S \) and \( T \). We further claim that \( z \) is the unique common fixed point of \( S \) and \( T \). If not, then let \( w \neq z \) be another common fixed point of \( S \) and \( T \), then \( d(w, z) > 0 \). Now, putting \( x = w, y = z \) in (3.1), we have
\[
d(Tw, Tz) \geq \lambda d(Sw, Sz) + \mu[d(Sw, Tz)+d(Sz, Tw)], \text{ i.e., } \]
\[d(w, z) \geq \lambda d(w, z) + \mu[d(w, z)+d(w, z)]; \text{ or, } [(1-\mu)/(\lambda+\mu)] \geq 1, \text{ (as } d(w, z) > 0)\]
a contradiction of \( [(1-\mu)/(\lambda+\mu)] < 1 \). Thus \( w = z \). Hence \( z \) is the unique common fixed point of \( S \) and \( T \). This completes the proof.

If \( S \) is an identity mapping, \( S = I \) then Theorem 3.1 reduces to following corollary:
Corollary 3.2 Let \((X,d)\) is a cone b-metric space with constant \(b \geq 1\). Suppose \(T:X \to X\) is a self-mappings, satisfying:
\[
d(Tx, Ty) \geq \lambda d(x, y) + \mu(d(x, Tx) + d(y, Ty)), \quad \forall x, y \in X, x \neq y,
\]
where \([(1-\mu)/(\lambda + \mu)] \in (0, 1/b),\) with \(\lambda + \mu > 0\) and \(\mu < 1\). If \(T\) is a surjection and \(T(X)\) is complete, then \(T\) has a unique common fixed point in \(X\).

This is a slight generalization of Theorem 2.1 of Xi-Wen [24] for cone b-metric space, instead of the cone metric space. Thus our Theorem 3.1 is a generalization of Theorem 2.1 of Xi-Wen [24] for two mappings in the cone b-metric space.

Interchanging \(S\) and \(T\) mutually, and taking \(\mu = 0\), we get the following corollary:

Corollary 3.3 Let \((X, d)\) be a cone b-metric space with constant \(b \geq 1\). Suppose \(T, S:X \to X\) are two self-mappings, satisfying:
\[
d(Sx, Sy) \geq \lambda d(Tx, Ty), \quad \forall x, y \in X, x \neq y,
\]
where \(\lambda > 1,\) \(1/\lambda \in (0, 1/b),\) (3.7)

If \(T(X)\) and one of the range subspaces \(S(X)\) or \(T(X)\) is complete, then \(S\) and \(T\) have a point of coincidence. Moreover, if the pair \((S, T)\) is weakly compatible, then \(S\) and \(T\) have a unique common fixed point in \(X\).

Here \(\lambda > b \geq 1\). Hence, Corollary 3.3 is a generalization of Theorem 2.3 of Xi-Wen [24] for cone b-metric space. Remark: Weakly compatibility of the pair \((S, T)\) is necessary in Theorem 3.1, as shown in following Example:

Example 3.4 Let \(E= C^1([0, 1], \mathbb{R}), \) \(P=\{\phi \in E: \phi(t) \geq 0, t \in [0, 1]\}\), \(X= [0, 1]\) and \(d: X \times X \to E\) defined by: \(d(x, y)=|x-y|^2\phi,\) where \(\phi \in P\) is a fixed function, \(e.g. \phi(t) = e^t.\) Then \((X, d)\) is a cone b-metric space. Since \(e^t \geq 1,\) for all \(t \in [0, 1]\), \(d\) is a b-metric with \(\phi(t) = e^t \geq 1.\) Also,
\[
d(x, y) = |x-y|^2\phi = (|x-z|+|z-y|)^2 \phi \leq \{ |x-z|^2 + 2|x-z| \phi \} \phi = \{ |x-z|^2 + |z-y|^2 + 2\sqrt{ |x-z| \phi |z-y|} \} \phi \leq \{ |x-z|^2 + |z-y|^2 \} \phi \quad \text{as} \ \forall ab \leq ½ (a+b) .
\]

Thus \((X, d)\) is a cone b-metric space with constant \(b=2.\) Define mappings \(S, T: X \to X\) by:
\[
Tx=1-x \quad \text{and} \quad Sx=1-(x/2), \quad \forall x \in X.
\]

Observe that \(S(X) = \{1/2, 1\} \subseteq T(X) = [0, 1].\) The pair \((S, T)\) has coincidence point \(x=0,\) where \(S\) and \(T\) have a unique common fixed point in \(X.\)

Remark: Weakly compatibility of the pair \((S, T)\) is necessary in Theorem 3.1, as shown in following Example:

Example 3.5 Let \((X, d)\) be a cone b-metric space as in Example 3.4.

Suppose \(S, T: X \to X\) are defined by: \(Sx=x/4,\) \(Tx=x/2\) for all \(x \in X= [0, 1].\) Then we observe that \(S(X) = \{0, 1/4\} \subseteq T(X) = \{0, 1/2\}.\) Pair \((S, T)\) is weakly compatible at \(u = 0\), where it has point of coincidence. \(x = 0\) is the only common fixed point of \(S\) and \(T.\) Here arises following cases:
Case-1 \( x\neq y \)

\[
d(Tx, Ty) = |x/2 - y/2|^{2\phi} = (1/4)d(x, y), \quad d(Sx, Sy) = |x/4 - y/4|^{2\phi} = (1/16)d(x, y), \quad d(Sy, Ty) = |y/2 - y/4|^{2\phi} = (1/16)d(y, 0).
\]

Hence from (3.1), we have

\[
d(Tx, Ty) = |x/2 – y/2|^{2\phi} = (1/4)d(x, y), \quad d(Sx, Sy) = |x/4 – y/4|^{2\phi} = (1/16)d(x, y), \quad d(Sy, Ty) = |y/2 – y/4|^{2\phi} = (1/16)d(y, 0).
\]

Therefore, we assume that \( y_{n+1} \neq y_n \), \( d(Tx, Ty) = 0 \). Hence from (3.1), we have

\[
d(Tx, Ty) = |x/2 – y/2|^{2\phi} = (1/4)d(x, y), \quad d(Sx, Sy) = |x/4 – y/4|^{2\phi} = (1/16)d(x, y), \quad d(Sy, Ty) = |y/2 – y/4|^{2\phi} = (1/16)d(y, 0).
\]

Hence from (3.1), we have

\[
d(Tx, Ty) = |x/2 – y/2|^{2\phi} = (1/4)d(x, y), \quad d(Sx, Sy) = |x/4 – y/4|^{2\phi} = (1/16)d(x, y), \quad d(Sy, Ty) = |y/2 – y/4|^{2\phi} = (1/16)d(y, 0).
\]

So, Theorem 3.1 is validated.

### B. Fixed point for T-extension condition for expansive mappings

Our second main result is following. We use the method of proving used in [24]:

**Theorem 3.6** Let \( (X, d) \) be a complete cone b-metric space with constant \( b \geq 1 \). Let \( S: X \to X \) be a continuous mapping satisfying the T-extension condition, satisfying:

\[
d(Tx, Ty) \geq \eta \ d(TSx, TSy), \quad \forall x, y \in X, \ x \neq y, \text{ where } \eta \geq b.
\]

where \( T: X \to X \) is an injective, continuous and sequentially convergent mapping. Then \( S \) has a unique fixed point \( z \in X \).

**Proof.** Let \( x_0 \) be an arbitrary point complete cone b-metric space. We construct two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \), defined by:

\[
x_0 = x_1, \quad x_1 = y_1, \quad x_2 = x_3, \quad \ldots, \quad x_{n+1} = Sx_n, \quad y_0 = y_1, \quad y_1 = T y_0, \quad y_2 = T y_1, \quad \ldots, \quad y_{n} = T y_{n-1}.
\]

Note that, if \( y_{r+1} = y_r \) for some \( r \in \mathbb{N} \), then \( T y_{r+1} = y_r \). Since \( T \) is an injective mapping, \( x_{r+1} = x_r \). Thus \( x_r \) is a fixed point of \( S \). Therefore, we assume that \( y_{m+1} = y_m, \forall n \in \mathbb{N} \). Putting \( x = x_m, \ y = y_n \) in (3.8), we have

\[
d(Tx_m, y_n) \geq \eta \ d(Sx_m, T y_n),
\]

\[
d(y_{m+1}, y_n) \geq \eta \ d(Tx_{m+1}, T y_{n+1}) = \eta \ d(y_{m+1}, y_{n+1}), \quad \forall \ n \geq m,
\]

or,

\[
d(y_{m+1}, y_n) \leq k \ d(y_m, y_n), \text{ where } k = 1/\eta, \text{ with } k \in (0, 1)
\]

(3.9)

Now, we show that \( \{y_n\} \) is a Cauchy sequence in \( X \). We precede method used in Lemma 2.3. By the simple induction with the condition (3.8), we have

\[
d(y_{m+1}, y_n) \leq k^2 \ d(y_{m+1}, y_{n-2}) \leq \ldots \leq k^m \ d(y_1, y_0).
\]

(3.10)

For any integers \( m \geq 1, p \geq 1 \), it follows that

\[
d(y_{mp+1}, y_n) \leq \theta \ d(y_{mp+1}, y_{mp}) + d(y_{mp}, y_{mp-1}) + \ldots + d(y_0, y_0)
\]

(4.9)

Let \( \theta \ll c \) be given. Note that \( b^p k^{p-1} d(y_1, y_0)/(b-k) + b^{p-1} k^m d(y_1, y_0) \to \theta \) as \( m \to \infty \) for any \( k \).
So, by Lemma 1.11, \( \{y_n\} \) is a Cauchy sequence in \((X, d)\). This completes the first part.

Further, since \(X\) is complete, there exists a point \( w \in X \) such that

\[
\lim_{n \to \infty} y_n = \lim_{n \to \infty} T x_n = \lim_{n \to \infty} T S^n x_0 = w
\]  

(3.11)

Since \(T\) is sequentially convergent, \(\{x_n\} = \{S^n x_0\}\) converges to some point in \(X\), say \(z\). By the continuity of \(T\), we have \(Tz = w\). Also, since \(TS\) is continuous, we have

\[
w = \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} T S^n x_0 = T S z.
\]  

(3.12)

We obtain \(T S z = T z\). Since \(T\) is injective, we get \(S z = z\). To conclude the proof, let us show that \(z\) is the unique fixed point of \(S\). Assume the contrary, that is, there exists \(w \in X\) such that \(w \neq z\) and \(S w = w\). Thus, \(d(z, w) > 0\) and since \(T\) is injective, we get \(d(Tz, Tw) > 0\).

Putting \(x = z\) and \(y = w\) in (3.8), we have

\[
d(Tz, Tw) \geq \eta d(TSz, TSw) = \eta d(Tz, Tw) \geq d(Tz, Tw), \quad \text{as } \eta > 1.
\]

which is a contradiction. Thus \(w = z\). This shows that \(z\) is a unique common fixed point of \(S\), where \(z = \lim_{n \to \infty} S^n x_0 = \lim_{n \to \infty} T^n x\). This completes the proof.

IV. CONCLUSION

Fixed point theory has many applications in modern mathematics. The cone metric spaces and b-metric spaces are generalizations of metric space. The cone b-metric space is a generalized form of both cone and b-metric spaces. In this paper, we have proved some common fixed point results for expansive type mappings in this space. The main part of the paper is to introduce the T-extension conditions in cone b-metric spaces. Suppose \(S: X \to X\) be a self-mapping defined in the cone b-metric space. Let \(S\) has a fixed point in \(X\), but it does not satisfy an expansive condition. Now, T-extensions mappings are those mappings, which make the given self-mapping \(S: X \to X\) to be an expansive type mappings so that the common fixed point can be extracted from \(S\) and \(T\). More precisely, there are some mappings, say \(S: X \to X\), which have a fixed point but do not satisfy an expansive condition. If another mapping \(T: X \to X\) is introduced, then expansive condition holds. In this case \(S\) is said to satisfy T-extension condition. In the second part, we will prove a common fixed point theorem for self-mapping \(S: X \to X\), with mapping \(T: X \to X\) under T-extensions satisfying an expansive type conditions in the cone b-metric space. In the first part of the paper, we have proved a common fixed point theorem for compatible mappings satisfying an expansive condition in this space. We have validated our theorems by examples.

REFERENCES


AUTHOR’S BIODATA

The author R. K. Verma is an Assistant professor in Mathematics in Chhattisgarh, India. He has teaching experience of about 22 years and research experience of nearly 15 years. He has 30 published papers. Prof. Dr. R. K.Verma is the author of a textbook “Differential Equation” for B.Sc.-II year students, and co-authored of seven textbooks of B.Sc. classes. All books are in Hindi medium.